G Chrystal
Edinburgh
April 1881

# PROBLEMS AND EXAMPLES.

PRINTED BY JONATHAN PALMER,

FOR

MACMILLAN AND CO. CAMBRIDGE.

London: BELL AND DALDY, 186, Fleet Street.
Oxford: J. H. AND JAS. PARKER.
Edinburgh: EDMONSTON AND DOUGLAS.
Glasgow: JAMES MACLEHOSE.
Dublin: WM. ROBERTSON.

# A COLLECTION OF

# PROBLEMS AND EXAMPLES

## IN MATHEMATICS

*SELECTED FROM THE JESUS COLLEGE EXAMINATION PAPERS*

## WITH ANSWERS.

BY

H. A. MORGAN, M.A.

SADLERIAN AND MATHEMATICAL LECTURER OF JESUS COLLEGE, CAMBRIDGE.

Cambridge:

MACMILLAN AND CO.

1858.

Gift of J Wangenheim
to Math. Dept.

TO THE

REV. PERCIVAL FROST,

MATHEMATICAL LECTURER OF JESUS COLLEGE, CAMBRIDGE,

&c., &c., &c.,

THESE PAGES

CONTAINING A LARGE NUMBER OF PROBLEMS

PROPOSED BY HIM DURING THE LAST TEN YEARS,

ARE DEDICATED,

AS A TOKEN OF ESTEEM

AND AS A TRIBUTE OF REGARD FOR

DISTINGUISHED TALENT.

# PREFACE.

THE Problems contained in the following pages have been selected from the Jesus College Papers, and nearly the whole of them have been proposed at the half-yearly examinations during the last ten years.

Those in Euclid, Conic Sections, Algebra, and Plane Trigonometry, have, with a few exceptions, been set during the latter half of this period.

It will be observed that some of them have been inserted in Mathematical books which have recently appeared, and that others are only particular cases of well-known theorems; but I believe that by far the larger number will be met with here for the first time.

I have been induced to publish them, in the hope, that a collection of Problems, a large proportion of which are easy, in the elements of most of the subjects usually read at Cambridge would be found useful.

Although the great majority of these Questions are due to the Lecturers of the College, papers have on several occasions been contributed by others, and I must not omit to mention the valuable aid which has thus been afforded by

E. Walker, Esq., late Fellow of Trinity College, the Rev. J. Wolstenholme, Fellow of Christ's College, and J. E. Prescott, Esq., late Fellow of Corpus Christi College.

The results have always been given, except in a few cases, in which, from the nature of the questions they would have been almost equivalent to solutions; in preparing them, the kind assistance of several friends has enabled me to render them far more free from errors than they otherwise would have been.

H. A. MORGAN.

JESUS COLLEGE,
*May 13th*, 1858.

# CONTENTS.

# ERRATA.

| Page | Problem | |
|---|---|---|
| 17 | 9 (2), | for 5 read 4 in both cases. |
| 24 | 56, | erase the term $f(x)$ in the second member of the equation. |
| 35 | 29, | for $2\alpha \cot\alpha$ read $2a \cot 2a$. |
| 37 | 45, | for cos read cot. |
| [illegible] | [illegible] | [illegible] |
| | | [illegible] $2y$ read $y$. |
| | | [illegible] ircle. |
| [illegible] | [illegible] | [illegible] |
| 131 | 19, | for shew that, read find how far. |
| 133 | 30, | for $n$ read $2n$. |
| — | 32, | for of $A$ &c. read of $C$ will be parallel to $BA$ if $\tan\alpha = \frac{2}{\sqrt{5}}$. |
| 138 | 30, | this result is incorrect, see answer to this question. |
| 157 | 15, | for 4200 read 4000. |
| 185 | 29, | for $\gamma$ read $V$. |

## ERRATUM.

Page 15. Problem 2, after rectum, insert $\times \frac{AC^2}{AB^2}$.

# ERRATA.

| Page | Problem | |
|---|---|---|
| 17 | 9 (2), | for 5 read 4 in both cases. |
| 24 | 56, | erase the term $f(x)$ in the second member of the equation. |
| 35 | 29, | for $2\alpha \cot\alpha$ read $2a \cot 2\alpha$. |
| 37 | 45, | for cos read cot. |
| 45 | 92, | for $\log\left\{\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$ read $\log\left\{\cot\frac{\theta}{2}.\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$. |
| 47 | 2, | for bisect read trisect. |
| 50 | 24, | in the first equation for $y$ read $2y$, and in the second for $2y$ read $y$. |
| 51 | 27, | the latter part of this equation is incorrect. |
| 65 | 8, | for $\omega_n$ read $u_n$. |
| 74 | 66, | for $a^2$ read $\alpha^2$. |
| 81 | 6, | for three straight lines read two straight lines and the circle. |
| 123 | 10, | for $\alpha$ read $\propto$. |
| — | 11, | for axis read base. |
| 125 | 21, | this result is incorrect; no neat result obtainable. |
| 131 | 19, | for shew that, read find how far. |
| 133 | 30, | for $n$ read $2n$. |
| — | 32, | for of $A$ &c. read of $C$ will be parallel to $BA$ if $\tan\alpha = \frac{2}{\sqrt{5}}$. |
| 138 | 30, | this result is incorrect, see answer to this question. |
| 157 | 15, | for 4200 read 4000. |
| 185 | 29, | for $\gamma$ read $V$. |

# PROBLEMS.

## EUCLID.

1. If any two of the sides of a parallelogram which meet each other make equal angles with either diameter, then the figure is a rhombus.

2. On the sides $AB$, $BC$, $CD$, $DA$, of a parallelogram set off $AE$, $BF$, $CG$, $DH$, equal to each other and join $AF$, $BG$, $CH$, $DE$, these lines form a parallelogram, and the difference of the angles $AFB$, $BGC$, equals the difference of any two proximate angles of the two parallelograms.

3. Draw lines through the angular points of a parallelogram which shall form another parallelogram equal to twice the former.

4. In the fig. Bk. 1 Prop. 7 shew that, if $AC=BD$ and $AD=BC$, then $CD$ is parallel to $AB$; but if $AC=BC$ and $AD=BD$, then $CD$ is perpendicular to $AB$.

Apply this proposition to shew that two circles cannot intersect in more than two points, one on each side of the line joining their centers.

5. If a straight line terminated by the sides of a triangle be bisected, no other line terminated by the same two sides can be bisected in the same point.

6. If two triangles have their areas equal, and one side and an adjacent angle equal in each, they shall be equal in all respects.

Hence shew that, if a parallelogram and a quadrilateral on the same base and between the same parallels be equal to one another, then the quadrilateral is a parallelogram.

7. Let $ACB$, $ADB$, be two right-angled triangles having a common hypothenuse $AB$, join $CD$, and on $CD$ produced both ways draw perpendiculars $AE$, $BF$. Shew that

$$CE^2 + CF^2 = DE^2 + DF^2.$$

8. From $AC$ the diagonal of a square $ABCD$ cut off $AE$ equal to one-fourth of $AC$, and join $BE$, $DE$. Shew that the figure $BADE$ equals twice the square on $AE$.

9. Let $ABCD$ be a rectangular parallelogram, $G$, $H$, two points in $AD$ and $DC$, and $E$ and $F$ in $AB$ and $BC$ respectively, such that $GH$ equals $EF$.

Join $AH$, $AF$, $CG$, $CE$, and shew that

$$AH^2 + CG^2 = AF^2 + CE^2.$$

10. In the fig. Bk. 1, Prop. 47, let $BAC$ be the right angle, $ABFG$, $BCED$, $CAHK$, the three squares and $AL$ perpendicular to $DE$. Prove that $AD$, $AE$, are at right angles to $FC$, $BK$, and that if $AD$, $AE$, $AL$, cut $BC$ in $M$, $N$, $R$, then

$$BM : CN :: BR : CR.$$

11. In the fig. Bk. 2, Prop. 9,

$$(1)\ AD^2 - DB^2 = 2AB.CD.$$

$$(2)\ AB^2 = 4CD^2 + 4AD.DB.$$

12. $ABCD$, $AECF$, are two parallelograms, $EA$, $AD$ being in a straight line. Let $FG$ drawn parallel to $AC$ meet $BA$ produced in $G$. Then the triangle $ABE$ equals the triangle $ADG$.

13. Any polygons whatsoever described about a circle are to one another as their perimeters.

14. The sum of the alternate angles of *any* hexagon inscribed in a circle is equal to four right angles.

15. Find where a paper triangle must be folded so that the vertex may fall upon a given point in the base. What limitation is necessary respecting the position of the said point in order that the crease may not cut the base?

16. Take any two points $G$ and $H$ in the diagonal $BD$ of the square $ABCD$ and join $AG$, $AH$. Then

$$AG^2 \backsim AH^2 = HG.(DH \backsim GB).$$

17. $ABCD$ is a square, $AC$ its diagonal, bisect $AD$ in $E$, join $BE$ cutting $AC$ in $F$, then will

$$\frac{\triangle AEF}{1} = \frac{\triangle CEF}{2} = \frac{\triangle ABE}{3} = \frac{\triangle BCF}{4}.$$

18. If two circles touch each other externally, and on the part of their common tangent intercepted between the points of contact as diameter a circle be described, it will touch the line joining their centers.

19. With three given points as centers, not lying in a straight line, describe three circles which shall have three common tangents.

20. In a circle describe a triangle having each of the angles at the base one-third of the vertical angle.

21. Shew that if $BAC$ be an angle standing on the circumference $BC$, then the sum of the angles in the segments $AB$, $AC$ is invariable wherever the point $A$ be taken, and is equal to $BAC$ together with two right angles.

22. $ABC$ is an isosceles triangle, $AB$ the base, $D$ a point in a straight line parallel to $AB$, $DE$ perpendicular to $AB$, prove that the difference of the triangles $ADC$, $AEC$, is a constant area.

23. If an octagon be capable of having a circle described about it, shew that the sums of the alternate angles are equal.

24. Shew that if angles be formed by lines drawn from the angles of a quadrilateral figure described about a circle to the centre, and any two adjacent angles of these be equal, then the other two will be equal also.

25. Describe an isosceles triangle having each of the angles at the base equal to one-eighth of the third angle.

26. Shew that the inscribed circle must always have a different center from the circumscribed circle unless the triangle be equilateral.

27. If circles be described upon the sides and hypothenuse of a right-angled triangle as diameters, then the segments cut off by the hypothenuse from the former shall be similar to those cut off by the sides from the latter.

28. What will be the form of the base of a pyramid whose sides consist of the greatest possible number of equal equilateral triangles?

29. $ACB$ is a right-angled triangle on a fixed hypothenuse $AB$, $AEC$, $BDC$ equilateral triangles on the sides $AC$, $BC$, which are bisected in $G$, $F$. Prove that the sum of the triangles $AFD$, $BEG$ is constant.

30. If from a circle a segment be cut containing an angle equal to that which it subtends at the center, then the chords containing that angle will always be together equal to the line drawn from their point of meeting to the bisection of the remaining segment.

31. Find a point without two concentric circles from which if tangents be drawn to the circles the one shall be double of the other.

32. Through a given point in the base of a triangle draw a line which meeting one of the sides, produced or not, shall form a triangle bearing a given ratio to the given triangle.

33. Divide a straight line into two parts, such that twice the rectangle contained by one of the parts and the line made up of the whole and the other part shall be equal to the square of the whole line.

34. Let $AB$ be the diameter of a circle, $CD$ a chord parallel to $AB$, and equal to one-half of it; join $AC$, and let the tangent at $B$ meet $AC$ produced in $E$, shew that $AE=2AB$. If $ED$ produced meets $AB$ in $F$, shew that $AB=3.BF$.

35. Let $ABCD$ be a parallelogram, bisect $AB$, $DC$ in $E$ and $F$, join $AF$, $BF$, $CE$, $DE$, cutting the two diagonals in $P$, $Q$, $R$, $S$. Shew that $PQRS$ is a parallelogram whose area is one-ninth of the parallelogram $ABCD$.

36. $B$ is the middle point of the arc of a segment $ABC$ less than a semicircle. Produce $AB$ to $D$, and draw $DC$ perpendicular to $BC$. Then the circle described about $BCD$ will pass through the intersection of the tangents at $A$ and $C$.

37. If a circle can be described cutting the sides $AB$, $BC$, $CA$ of a triangle in the points $D, E$; $F, G$; $H, K$, in such a manner that $AD=BF=CH$, and $EB=GC=KA$, shew that the triangle is equilateral.

38. In the fig. Bk. 2, Prop. 13, Case 2, draw a perpendicular $CE$ from the obtuse angle $C$ upon the side $AB$, and prove that

$$AB^2=AB.AE+BC.BD.$$

39. Make an isosceles triangle of given altitude whose sides shall pass through two given points and have its base in a given straight line.

40. If the circumference of one circle passes through the center of another, any two chords of the second drawn from the points of intersection so as to cut one another in the said circumference will be equal.

41. The circumference of one circle passes through the centre $O$ of another; and through $A$ one of the points of intersection, a diameter $AB$ is drawn to the first meeting the other in $C$; shew that

$$AB.AC=2OC^2.$$

42. If two unequal circles cut one another; and the straight line, drawn through one of the points of intersection to the extremity of the diameter through the centers, be bisected in that point, then the circumference of the lesser circle will bisect the distance between the centers.

43. If lines be drawn through the angles of a parallelogram so as to form a second parallelogram, the segments which these taken in order produced or not, cut off from the sides taken in order of the first produced or not, shall be proportionals.

44. In any triangle the line bisecting an angle, and the line bisecting and perpendicular to the opposite side intersect in a point in the circumscribing circle.

45. If the interior and exterior angles at the vertex of a triangle be bisected by lines which cut the base and the base produced, then the sum of the segments of the base between the points of section and either of the other angles of the triangle is to their difference as the sides of the triangle are to each other.

46. If three circles touch each other in any manner, the tangents at the points of contact pass through the same point.

47. If three lines $MN$, $PQ$, $RS$, be drawn through any point within a triangle parallel to the sides $AB$, $AC$, $BC$, respectively, then will

$$\frac{MN}{AB}+\frac{PQ}{AC}+\frac{RS}{BC}=2.$$

48. Given the perimeter, the vertical angle and the perpendicular from one extremity of the base upon the opposite side of a triangle, construct the triangle.

49. If through two given points in the circumference of a circle, pairs of equal chords be drawn, one set of their intersections will lie in a diameter of the circle and the other in the circumference of a second circle, passing through the given points.

50. If $F$ be a point in the side $CB$ of a right-angled triangle $CD$, $FE$, perpendiculars on the hypothenuse $AB$, then

$$AD.AE+CD.EF=AC^2.$$

51. If a quadrilateral circumscribe a circle, its diagonals, and the lines joining the points of contact of opposite sides, meet in a point.

52. In a quadrilateral figure $ABCD$ is inscribed a second quadrilateral by joining the middle points of its adjacent sides; a third is similarly inscribed on the second, and so on. Shew that each of the series of quadrilaterals will be capable

of being inscribed in a circle if the first three are so. Shew also that two at least of the opposite sides of $ABCD$ must be equal, and that the two squares upon these sides are together equal to the sum of the squares upon the other two.

53. Let $ABCD$, $EFGH$ be parallelograms on the equal bases $BC$, $FG$, and between the same parallels. Let $BE$, $CH$, cut $DC$ and $EF$ in $K$ and $L$. Then if the points $A$, $K$, $L$, $G$, be in a straight line the parallelograms must be equiangular and

$$BF.FG = BG.CF.$$

54. $ABC$ is an equilateral triangle. Produce $AB$ to $D$ and make $BD$ equal to $AB$. With centre $D$ and radius $DC$ describe a circle. Then if any point $E$ be taken in the circumference and joined with $A$ and $B$, prove that

$$AB^2 + AE^2 = 2BE^2.$$

55. If the chord, or the chord produced, of one circle be a tangent to another, and perpendiculars be drawn from its extremities to the line passing through the centers; then the difference of the squares of the segments of the chord is equal to twice the rectangle contained by the line between the feet of the perpendiculars and the distance between the centers.

# CONIC SECTIONS.

## *Parabola.*

1. IF $AN$ be the abscissa of $P$, shew that

$$PY^2 = AN.PS.$$

Shew also that if $YK$ be the perpendicular dropt from $Y$ on $SP$, then

$$PN = 2.YK.$$

2. If $P$ be a point in a parabola, and a circle be described touching the axis of the parabola and the focal distance $SP$ in $P$; then the portion of the diameter through $P$ which is cut off by this circle is equal to the latus-rectum.

3. $AP$ is a parabola, $PD$ parallel to the axis meets the directrix in $D$, $PT$ the tangent meets the axis in $T$, shew that $SD$ bisects $PT$.

4. If the normal at $P$ meets the axis in $G$, and the ordinate at $G$ meets the parabola in $Q$ and $AP$ produced in $R$, then

$$QG^2 = PN.RG.$$

5. A circle has its center in the vertex $A$ of a parabola whose focus is $S$, the diameter of the circle is $3AS$, shew that the common chord bisects $AS$.

6. $PM$ is an ordinate of a point $P$ in a parabola, $QR$ is a diameter bisecting $PM$ and cutting the curve in $Q$, $MQ$ cuts the tangent at the vertex $A$ in $T$, shew that

$$AT = \tfrac{2}{3}PM.$$

7. Shew that in the parabola

$$\sin^2 PTS \propto \frac{1}{SP}.$$

8. If in the parabola, the perpendicular drawn from one extremity $Q$ of an ordinate meets $PV$ the diameter to that ordinate in $D$, then

$$DV^2 = 4AN.PV.$$

9. If a circle be described touching a parabola in the point $P$ and its axis in the focus, then the normal at $P$ will make with the axis an angle equal to $\frac{1}{3}$ a right angle.

10. A circle is described touching the axis of a parabola at the point where it meets the directrix, and a line perpendicular to the axis touching the circle in $R$, meets the parabola in $P$ and $Q$, and the axis in $N$; shew that

$$SN^2 = PR.QR.$$

11. A normal is drawn to a parabola at a point whose co-ordinates are each equal to the latus rectum, shew that it meets the parabola at a point whose distance from the axis is three times the half latus rectum.

12. Through any point $P$ of a parabola a straight line $QPQ'$ is drawn parallel to the latus rectum and terminated by the tangents at its extremities, and $PR$ is drawn perpendicular to the latus rectum produced if necessary; shew that

$$PR^2 = QP.PQ'.$$

13. If the normal $PG$ at a point $P$ of a parabola meet the directrix in $F$, shew that $PF \propto PS^{\frac{3}{2}}$, and if the perpendicular from $S$ meets the normal in $K$, then

$$FK.AS = SY.AG.$$

14. If in the parabola the tangents at the extremities of the focal chord $PSp$ meet the tangent at the vertex in the points $Y$, $Z$, shew that

$$YZ^2 = AS.Pp.$$

15. $PT$ is a tangent to a parabola at $P$, $SAT$ being the axis. On $AT$, as diameter, a circle is described, and with center $S$, a circle touching $PT$, the two circles intersect at right angles.

16. The focus of a parabola is given, and a circle, whose center is in the axis of the parabola, give a geometrical construction for the parabola touching the circle.

17. Shew that any parabola may be cut from any cone.

18. In the parabola, if $PSp$ be a focal chord,

$$SP^2 = Pp.AN.$$

19. A parabola, whose focus is given, touches two perpendicular straight lines: give a geometrical construction for the axis and vertex.

20. Shew that if a pair of tangents from a point $Q$ meet the tangent at the vertex in $Y$ and $Y'$, then $SY$, $SY'$ will be tangents to a parabola having $Q$ for its focus and $YY'$ for the tangent at the vertex.

21. If rhombi, having one corner common to all, and an adjacent side given in position, be described in such a manner that the intersections of their diagonals are always in a straight line perpendicular to the side given in position, then the locus of one of the other angles will be a parabola.

22. A circle touches a parabola at the extremities of a double ordinate $PNp$, and cuts its axis in $M'$, $M$. Shew that the difference of the ordinates at $M'$, $M$ and their sum are respectively equal to the latus rectum and to the diameter of the circle, and that the ordinate $PN$ is a mean proportional between these ordinates.

23. A circle is described on the latus rectum of a parabola as diameter, and through the focus is drawn a straight line meeting the circle and parabola in $P$, $Q$ respectively, shew that the tangents at $P$ and $Q$ intersect in the latus rectum, or else in a line parallel to it at a distance from it equal to the latus rectum.

24. With the focus $S$ of a parabola as centre, circles are described touching the tangents at the extremities of a focal chord $Pp$, and cutting $SP$, $Sp$ in $B$, $b$. $BE$, $be$ are drawn parallel to the normals at $P$, $p$ to meet the axis in $E$, $e$.

Shew that the square described on $AS$ is a mean proportional between the triangles $SBE$, $Sbe$.

Shew also that if from $G$ the foot of the normal a perpendicular $GL$ be drawn on $SP$, the triangle $PLG$ is double of the triangle $SBE$, and that each varies as the ordinate at $P$.

25. Two parabolas have a common vertex $A$ and axis; an ordinate $NPQ$ meets them in $P$ and $Q$; a tangent at $P$ meets the other parabola in $R$, $R'$; $AR$, $AR'$ meet the ordinate in $L$, $M$. Prove that $NP$, $NQ$ are respectively Harmonic and Geometric means between $NL$ and $NM$.

26. In a parabola $QV$ is the semi-ordinate to the diameter $PV$, and $QF$ is perpendicular to the ordinate $PR$. Join $QR$ and $VF$, then the triangles $PVF$, $FQR$ are similar.

27. $PSQ$ is a focal chord of a parabola whose vertex is $A$. If through $P$ and $Q$ lines be drawn perpendicular to the axis, and meeting $AQ$, $AP$ in $p$ and $q$, then

$$Pp.Qq=4AS.PQ.$$

---

## *Ellipse.*

1. If in an ellipse the circles described on $SY$, $HZ$ as diameters cut $SP$, $HP$, in $K$ and $L$ respectively, then $KL$ will be parallel to the major axis.

2. If any number of ellipses be described with the same minor axis and be cut by a common ordinate to that axis, the tangents at the points where the common ordinate cuts the curve will all pass through the same point.

3. $PM$, $PT$ are an ordinate and tangent at $P$ in an ellipse, shew that the circles whose diameters are $MT$ and the major axis intersect at right angles.

4. If the tangent at the extremity of the latus rectum meets the major and minor axes in $T$ and $t$, and a circle be described about the triangle $STt$, then the circle will touch the minor axis in $t$.

5. If the tangent at $P$ meets the axis major in $T$, then

$$TY.TZ-PY.PZ=PT^2.$$

6. If the base of a semicircle be divided into any two parts, and semicircles be described on them, the area included between the three circumferences is equal to an ellipse, whose axes are the segments of the said base. Find also the proportion it bears to the circles described on the said segments.

7. If from $B$ the extremity of the minor axis of an ellipse two straight lines $BK$, $BL$, be drawn parallel to $CP$, $CD$, two semiconjugate diameters, and meeting the major axis in $KL$, then

$$CK.CL=CA^2.$$

8. If through the foot $G$ of the normal at any point of an ellipse $GU$ be drawn parallel to the tangent at $P$, to meet the focal distance in $U$, then $PU$ is an harmonic mean between the focal distances.

9. If the ordinate at $P$ meet the auxiliary circle in $Q$, and the tangent to the circle at $Q$ meet the axis minor produced in $R$, then

$$RC.PN=AC.BC.$$

10. If in an ellipse $\rho_1$, $\rho_2$ be the radii of the circles touching the sides $SP$, $HP$ respectively, and the other two sides produced of the triangle $SPH$, and $\rho_3$ the radius of the other escribed circle, shew that

$$AS.\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)=HS.\frac{1}{\rho_3}.$$

11. Prove that

$$\cos\frac{SPH}{2}=\frac{BC}{DC}.$$

12. *CP*, *CD* are semiconjugate diameters to an ellipse; circles are described with a center *C* and radii *CP*, *CD*, cutting the axes *Aa*, *Bb* in *Q*, *R* respectively, shew that the rectangle *AQ*, *aQ* is equal to the rectangle *BR*, *bR*.

13. Two equal and concentric ellipses have their major axes at right angles to each other. If $A_1$ be the area of the circle passing through the points of intersection of the ellipses, and $A_2$ that of the circle passing through their foci, prove that

$$\frac{A_2}{A_1} = \tfrac{1}{2}\left(\frac{AC^2}{BC^2} - \frac{BC^2}{AC^2}\right).$$

14. Let the auxiliary circle to an ellipse meet the minor axis produced in *K*. Join *KT* and let it meet an ordinate *QPN* to the ellipse and circle in *R*. Then

$$RN : PN :: QN : BC.$$

15. The ordinates at *P* and *D* meet the auxiliary circle in *K* and *L*. Shew that the perpendiculars dropped from *P* and *D* on *CK*, *CL* respectively, are equal.

16. Through *P* and *D* the extremities of two semiconjugate diameters in an ellipse, *PK*, *DL* are drawn parallel to the axis major meeting *CD* and *CP* in *K* and *L*; shew that

$$\frac{1}{PK^2} + \frac{1}{DL^2} = \frac{1}{AC^2}.$$

17. A circle is described on the minor axis of an ellipse, *CD*, *CP* are half-conjugate diameters; shew that the tangent *DU* to the circle is parallel to *SP*.

18. In an ellipse the tangents at *P* and *D* meet *CA* and *CB* produced in points *T* and *K* respectively, prove that *TK* is parallel to *AB*.

19. *TP* is a tangent at *P* in an ellipse and meets the major axis in *T*, a circle described round *SPH* cuts *TP* in *t*, shew that

$$CD^2 = TP.tP.$$

20. The focal distance $SP$ of a point in an ellipse is produced to $Q$ until $QP$ is equal to the semi-major axis, $DCD'$ is the diameter conjugate to $CP$. Prove that the triangle $QDD'$ is to the area of the ellipse as $2 : \pi$.

21. The normals at $P$ and $D$, the extremities of the conjugate axes of an ellipse, intersect the major and minor axes in $G, K, G', K'$. Prove that the triangle $CGK =$ the triangle $CG'K'$.

22. With foci $S$, $H$ of an ellipse as centers, circles are described touching the tangent at any point $P$ of the ellipse and cutting $SP$, $HP$ in $D$, $d$; $DE$, $de$ parallel to the normal at $P$ meet the major axis in $E$, $e$. Prove that $DE = de =$ semi-latus rectum, and that $SE.He$ is invariable.

23. If $S$, $H$ be the foci of an ellipse, $P$ any point in it, and a circle described about the triangle $SPH$ meet the minor axis in $t$, $g$, shew that $Pt$, $Pg$ are respectively tangent and normal at $P$; and if $Pt$, $Pg$ meet the major axis in $T$, $G$, shew that

$$PT.Pt = PG.Pg = SP.HP.$$

24. The tangent at a point $P$ in an ellipse meets the axes in $T$ and $t$. Join $St$. Then the angle $PSt =$ angle $STP$.

25. Two similar ellipses have their axes parallel, and the center of one is in the circumference of the second, shew, without analysis, that the common chord is parallel to the tangent to the second at the center of the first.

26. If in an ellipse, $P$ and $D$ be the extremities of any two semi-conjugate diameters, and $B$ that of the minor axis, then

$$\tan^2 \frac{SPH}{2} + \tan^2 \frac{SDH}{2} = \tan^2 \frac{SBH}{2}.$$

27. In an ellipse, let the tangents at any points $P$ and $Q$ meet in $T$, and let $ST$ and $HT$ cut $PQ$ in $K$ and $L$.

Prove that $$\frac{SQ}{KQ} + \frac{HP}{LP} = 4\frac{AC}{PQ}.$$

28. Through any point $P$ of an ellipse lines are drawn parallel to equal conjugate diameters to meet the major axis in $K$, $L$, and the minor axis in $R$, $V$. Shew that

$$KR^2 + LV^2 = 2.(AC^2 + BC^2).$$

29. Given the foci and major axis of an ellipse, obtain by a geometrical construction the points in which the ellipse meets a given straight line drawn through one of the foci.

---

## *Hyperbola.*

1. The tangent at any point $P$ of a rectangular hyperbola meets the asymptotes in $Q$ and $R$. If $G$ be the foot of the normal at $P$, shew that $P$ is the centre of the circle circumscribing the quadrilateral $CQGR$.

2. If through the vertex of an hyperbola, a line be drawn perpendicular to one asymptote and meeting the other in $Q$ and the curve in $P$, the difference between the abscissæ of the points $P$ and $Q$ is equal to the semi-latus rectum.

3. From $G$, the foot of the normal, at any point $P$ of a rectangular hyperbola, $GE$ is drawn perpendicular to $CP$ produced. Prove that $PE$ is equal to the perpendicular from $P$ on the semi-conjugate diameter $CD$.

4. In the rectangular hyperbola, the perpendiculars drawn from the foci on the tangent at the extremity of the latus rectum are to one another as

$$3 : 1.$$

5. If the ordinate at any point $P$ in a rectangular hyperbola whose center is $C$ meet the asymptote in $Q$, then $CQ$ is an arithmetic mean between $SP$ and $HP$.

6. Let $H$ and $S$ be the foci of an hyperbola, $P$ a point in the curve, and $K$ the point of intersection of the tangents at $A$ and $P$. If now $H$ and $P$ be made the foci of another hyperbola which passes through $S$, then will $SK$ be a tangent to it; and the foot of a perpendicular from $K$ upon $HP$ will give its vertex.

## ALGEBRA.

1. PROVE that

(1) $$\frac{a+b}{ab}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{b+c}{bc}\left(\frac{1}{c}-\frac{1}{b}\right)=\frac{a+c}{ac}\left(\frac{1}{a}-\frac{1}{c}\right).$$

(2) $$(a^2+b^2)\div\left(\frac{1}{b}-\frac{1}{a}\right)-(a^2-b^2)\div\left(\frac{1}{b}+\frac{1}{a}\right)=\frac{2a^2b^2}{a-b}.$$

(3) $$\frac{a^2+b^2+c^2+d^2}{(ab-cd)^2+(ad+bc)^2}=\frac{1}{a^2+c^2}+\frac{1}{b^2+d^2}.$$

(4) $$\frac{bc}{a(a^2-b^2)(a^2-c^2)}+\frac{ac}{b(b^2-a^2)(b^2-c^2)}$$
$$+\frac{ab}{c(c^2-b^2)(c^2-a^2)}=\frac{1}{abc}.$$

2. Reduce to their lowest terms:

(1) $$\frac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}.$$

(2) $$\frac{e^{x-y}+xy^{-1}+yx^{-1}+e^{y-x}}{xy^{-1}e^{x-y}+2+yx^{-1}e^{y-x}}.$$

3. Prove that

(1) $$\left\{y\sqrt{(-1)}+\frac{1}{x\sqrt{(-1)}}\right\}^2-\left\{x\sqrt{(-1)}+\frac{1}{y\sqrt{(-1)}}\right\}^2$$
$$=(x^2-y^2)\left(\frac{1}{x}\cdot\frac{1}{y}-1\right)^2.$$

(2) $\frac{a^2+b^2}{a^2-b^2}-1\div\left(\frac{2a}{2a-b}-\frac{2b}{2b-a}\right)=\frac{5ab}{2(a^2-b^2)}.$

(3) $n.(n-1)(n-2)-p(p-1)(p-2)$
$=(n-p)\{(n+p-1)(n+p-2)-np\}.$

4. If $a:b::b:c::c:d$, shew that,

(1) $(b+c)(b+d)=(c+a)(c+d)$.

(2) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$.

(3) $\left(\frac{a-b}{c}+\frac{a-c}{b}\right)^2-\left(\frac{d-b}{c}+\frac{d-c}{b}\right)^2=(a-d)^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right).$

5. If $a:b::c:d::e:f$, then

(1) $\frac{a(a+mc)+nec}{b(b+md)+nfd}=\frac{a^2}{b^2}.$

(2) $(a^2+b^2)(ce+df)^2=(c^2+d^2)(ae+bf)^2=(e^2+f^2)(ac+bd)^2.$

6. If $a:b::b:c::c:d::d:e$ &c.,
then $b(a-b)+d(c-d)+$ &c. $:b(b-c)+d(d-e)+$ &c. $::a:b$.

7. If the ratio of $a+x:a-x$ equals the duplicate ratio of $a+b:a-b$, then

$$x-b:a-x::b(a+b):a(a-b).$$

8. If $x$ be to $y$ in the duplicate ratio of $m$ to $n$, and $m$ to $n$ in the subduplicate ratio of $p^2+x^2$ to $p^2-y^2$, shew that

$$p^2:xy::x+y:x-y.$$

9. Solve the following equations:

(1) $x^3+8=2x^2+11x+14$.

(2) $x^4+5-2\sqrt{(x^2-2)}=5x^2$.

(3) $(x+1)(x+3)+(x+2)^2=x^2(x+4)^2-8$.

(4) $2\sqrt{(1-x^2)}+\sqrt{(1+x)}+\sqrt{(1-x)}=x-\frac{5}{4}$.

(5) $$\left.\begin{aligned} xy+\frac{x}{y}&=\frac{5}{3},\\ \frac{1}{xy}+\frac{y}{x}&=\frac{20}{3}.\end{aligned}\right\}$$

(6) $$\left.\begin{aligned} \frac{x^3}{y}-\frac{y^3}{x}&=\frac{15}{2},\\ \frac{x}{y}-\frac{y}{x}&=\frac{3}{2}.\end{aligned}\right\}$$

(7) $$\left.\begin{aligned} \sqrt{(x^2+a^2)}+\sqrt{(y^2+b^2)}&=\sqrt{\{(x+b)^2+(y+a)^2\}},\\ \left(1+\frac{b}{x}\right)\left(1+\frac{a}{y}\right)&=4.\end{aligned}\right\}.$$

(8) $$\left.\begin{aligned} x+y+z&=7,\\ xy+xz&=yz-2,\\ x^2+y^2+z^2&=21.\end{aligned}\right\}$$

(9) $$\left.\begin{aligned} x\sqrt{\frac{x}{y}}+y\sqrt{\frac{y}{x}}&=34,\\ x-y&=12.\end{aligned}\right\}$$

(10) $$\left.\begin{aligned} x^2y-y^2x+20\,(x^2+y^2)&=0,\\ y^2-20x^2&=xy.\end{aligned}\right\}.$$

(11) $$\left.\begin{aligned} a^3\frac{x}{y}-y&=a,\\ b^3\frac{y}{x}-x&=b.\end{aligned}\right\}.$$

(12) $$\left.\begin{aligned} 8y^4-9y^3+16xy^2+8x^2&=0,\\ 2y^2-5y+2x\phantom{{}+8x^2}&=0.\end{aligned}\right\}.$$

(13) $$2a\sqrt{x}+\sqrt{(x+\sqrt{x})}=ax\,(a^2-1).$$

(14) $$\sqrt{\frac{x}{y}}(2x-7y)+\sqrt{\frac{y}{x}}(2y-7x)=4\sqrt{(x^2+xy+y^2)},$$
$$2x-15y=221.$$

10. *A* and *B* walk over the same ground going out one way and coming home the other, but they start in opposite directions. *A* walks $3\frac{3}{4}$ miles per hour and *B* walks 4 miles per hour. *A* wants $\frac{1}{4}$ of a mile of being half-way when he meets *B*. Required the length of the walk.

11. There are two purses of which the first contains three times as many half-crowns as the other crowns, and the

sum in the first is as much over £10 as that in the other is under it.

Find the sum in each.

12. A clock gains $3\frac{1}{4}$ minutes in 15 seconds under the twenty-four hours. At noon it is 2 minutes too slow, when will it indicate the correct time?

13. A ship 40 miles from the shore springs a leak which admits $3\frac{3}{4}$ tons of water in 12 minutes. 60 tons would suffice to sink her, but the ship's pumps can throw out 12 tons of water in an hour. Find the average rate of sailing that she may reach the shore just as she begins to sink.

14. In the examinations for appointments to the Indian Civil Service there are given for English literature 1500 marks, for Classical literature 1500, for Modern Languages 1125, for Mathematics 1000, for Natural Sciences 500, for Moral Sciences 500, for Eastern Languages 750.

Two candidates compete, whose proficiences in these several branches are as the numbers 2, 3, 1, 3, 2.5, 3, 1, and 1.5, 2.5, 3, 3, 4, 3, 0, respectively. Which will stand first?

15. $A$ is twice, and $B$ just one and a-half times as good a workman as $C$. The three work together for two days, and then $A$ works on alone for half a day. How long would it have taken $A$ and $C$ together to complete as much as the three will have thus performed?

16. Two men $A$ and $B$ agree to perform two equal pieces of work. $A$ begins one piece and $B$ the other, and after working for $p$ days they change pieces. $A$ finishes his piece in $q$ days and then returns, and helps $B$, and after working together for $r$ days they finish his piece. How long would each be in doing either piece?

If $A$ is twice as good a workman as $B$, shew that $q = p + 3r$.

17. At noon on a certain day two watches indicate, the first true time, the second ten minutes past One. The first having been stopped for four hours in the interval, at noon next day points to the same time as the second, which loses one hour in twenty-four. Find the rate of gain of the first watch.

18. If the ratio of the roots of the equation
$$x^2 + px + q = 0$$
be equal to that of the roots of
$$x^2 + p_1 x + q_1 = 0,$$
then
$$p^2 . q_1 = p_1^2 . q.$$

19. Solve the equation $x^3 (ax + b) - (a + bx) = 0$, and if $s$ be the sum, and $p$ the product of the roots, shew that
$$as - bp = 0.$$

20. If $a$, $b$, $c$, $d$ be in geometric progression, then will
$$(a - d)^2 = (b - c)^2 + (c - a)^2 + (d - b)^2.$$

Shew also that if $\alpha^2$, $\beta^2$, and $\gamma^2$ be in arithmetic progression, then will
$$\frac{1}{\beta + \gamma}, \quad \frac{1}{\gamma + \alpha}, \quad \text{and} \quad \frac{1}{\alpha + \beta}$$
be in arithmetic progression.

21. If $a$, $b$, $c$, be in harmonic progression, then the harmonic mean between $a$ and $b$, $b$, and the harmonic mean between $b$ and $c$, will also be in harmonic progression.

22. If $S$ be the sum of the squares, and $S'$ the square of the sum of a decreasing geometrical progression continued to infinity, $r$ being the common ratio, prove that
$$S(r + 1) + S'(r - 1) = 0.$$

23. If there be two arithmetical progressions such that any two given terms in the one are proportional to two similarly situated terms in the other, shew that this will be always the case.

24. If $(ab + bc + 2ac)(ac - bc) = a^2b^2 - b^2c^2$, then $a$, $b$, $c$, are in harmonic progression.

25. If $a$, $b$, $c$, be in arithmetic, and $a$, $b$, $d$, in harmonic progression, prove that
$$\frac{c}{d} = 1 - \frac{2(a - b)^2}{ab}.$$

26. If $a$, $b$, and $c$, be in harmonic progression, shew that

$$b^2(a-c)^2=2\{c^2(b-a)^2+a^2(c-b)^2\}.$$

27. Prove that

$$(a+c^{-1})^{\frac{1}{2}}+(c+a^{-1})^{\frac{1}{2}}>2^{\frac{3}{2}};$$

28. Shew that if $\dfrac{a^2}{2b\sqrt{(2a^2-b^2)}}<1$, $a$ must be $<b(1+\sqrt{3})$.

29. Shew that if $A_1$, $H_1$ be the arithmetic and harmonic means between two quantities, and $A_2$, $H_2$ those between any two others, then

$$A_1H_1+A_2H_2>2H_1H_2.$$

30. If $ad=bc$, shew that

$$(a^n+b^n+c^n+d^n)(a+b+c+d)^n=\{(a+b)^n+(c+d)^n\}$$
$$\{(a+c)^n+(b+d)^n\}.$$

31. If $\dfrac{a-b}{ay+bx}=\dfrac{b-c}{bz+cz}=\dfrac{c-a}{cy+az}=\dfrac{a+b+c}{ax+by+cz}$; then each of these fractions will equal

$$\frac{1}{x+y+z}.$$

32. On a railway there are 20 stations, find the number of tickets required in order that a person may travel from any one station to any other.

33. A pack of cards is dealt into four heaps. How many different hands of four may be made by taking one out of each heap?

34. There are ten soldiers and eight sailors. How many different parties of six can be made, each party consisting of three soldiers and three sailors?

35. Five four-oars have to be made up, the number of men is exact, but five of them can only steer. In how many ways can the crews be chosen?

36. A train consists of 12 carriages, of which 5 are first class, 4 second, and the rest third class. In how many

different ways may the carriages be arranged? In how many different ways may they be arranged so that all the first class may be together?

37. If $p_2, p_3, \ldots p_n$ represent the number of permutations that can be formed out of $n$ quantities, taken 2, 3,...$n$ together respectively, and $P = p_2, p_3 \ldots p_n$, shew that

$$P = p_3 . p_{n-1} . \{(p_3 - p_2)(p_4 - p_3) \cdots (p_{n-1} - p_{n-2})\}.$$

38. If $r$ be the number of things which must be taken out of a given even number $n$ in order that the number of combinations formed by them may be the greatest possible, prove that the number of combinations $r$ together : the number of permutations of $r$ things all together

$$:: 2^{\frac{3n}{2}} . 1.3.5 \ldots . (n-1) : (2.4.6 \ldots n)^2.$$

39. If $\dfrac{a+cx}{c+ax}$ be expanded in series ascending by powers of $(1-x)$ and $(1+x)$, and $A$ and $B$ be the coefficients of $(1-x)^n$ and $(1+x)^n$ respectively; then

$$\frac{A}{B} = \pm \left(\frac{c-a}{c+a}\right)^{n+1},$$

the upper or lower sign being taken according as $n$ is even or odd.

40. The intervals, by which the convergents that are greater than the true value of a fraction approach that value, are given at any period by dividing the "quotient" by the product of the denominators of the corresponding convergent and the greater next preceding it. And similarly for the convergents that are less than the true value.

41. In the scale of $r$, the square of a number consisting of $r$ units will have cyphers in the $r^{\text{th}}$ and $(r+1)^{\text{th}}$ places, reckoning from the right hand, and the highest digit of the scale in the $(r-1)^{\text{th}}$ and $(r+2)^{\text{th}}$ places.

42. Prove that the sum of the cubes of three even numbers in arithmetic progression is divisible by 24.

43. If 5 be subtracted from the sum of the squares of any two consecutive numbers each prime to 3, the remainder will be divisible by 36.

44. If $p$, $q$, and $r$ be three consecutive primes to 3, prove that

$$p(p-2q)-r(r-2q)=\pm 3,$$

the upper or lower sign being taken according as $q$ exceeds $p$ by 2 or 1.

45. If $m$ and $n$ be two prime numbers, shew that the sum of all the numbers less than $mn$ and prime to it is

$$\frac{mn.(m-1)(n-1)}{2}-1.$$

46. If $e$ be the base of the Napierian system of logarithms and $x$ be small, shew that

$$e^{e^x}=e(1+x), \text{ nearly.}$$

47. Find $x$ and $y$ from the equations

$$(ax)^{\log a}=(cy)^{\log c},$$
$$c^{\log x}=a^{\log y}.$$

48. Shew that

$$\log_{10}1234321=6+2M(.1+\frac{.01}{2}+\frac{.001}{3}+\&c.-.0001-\&c.)$$

where $M$ is the modulus.

49. A person insures his life for £2000, paying an annual premium of £22 16*s*. Shew that if he should live 39 years it would have been more profitable for him to have invested the premium annually at 4 per cent. compound interest.

mant. 104 = 0170333,
mant. 257 = 4099331,
mant. 57 = 7558749.

50. Find all the positive integral solutions of the equations

$$y(x-3)=x^2,$$
$$x-y+xy=9.$$

51. The distance between two poles is known to be between 300 and 600 yards, and a person when measuring the distance with a chain 40 yards long finds there are 14 yards over, but when he measures it with a chain 32 yards long he finds there are 6 yards over. Find the distance between the poles.

52. Four boats start in a race, their respective velocities, considered uniform, being in the proportions of 1, $1+r$, $1+3r$, $1+5r$; where $r$ is less than 1. At the end of five minutes a bump takes place, and at the end of three minutes more another. Shew that the distance between the boats at starting was twice the length of one of them.

53. Eliminate $x$, $y$, and $z$, from the equations

$$\left.\begin{aligned} \frac{x^4}{a^{\frac{5}{4}}b^{\frac{3}{4}}} + \frac{y^4}{a^{\frac{3}{4}}b^{\frac{5}{4}}} &= mz^2, \\ \frac{x^{\frac{3}{2}}}{(az)^{\frac{1}{4}}} = x + y &= \frac{y^{\frac{3}{2}}}{(bz)^{\frac{1}{4}}} \end{aligned}\right\},$$

and shew that if $\frac{a}{b}$ is possible, $m$ cannot be less than $2^9$.

54. If ${}^nC_r$ be the coefficients of $x^r$ in the expansion of $(1+x)^n$, and ${}^nC_r'$ that of $x^r$ in $(1-x)^{-n}$, prove that

$$\frac{n+r}{n} \cdot \frac{\lfloor r}{(r+1)(r+2)\ldots 2r} \cdot {}^nC_r\,{}^nC_r' = {}^{n+r}C_{2r}.$$

55. Prove that if $n$ be a positive integer

$$(1+x)^n(1+x^n) > 2^{n+1}x^n.$$

56. If $f(x) = e^x - 1$, and $\phi(x) = e^x + 1$, shew that

$$f(x) . \log \tfrac{1}{2}[f\{\phi(x)\} + \phi . \{\phi(x)\}]$$
$$= \phi(x) . \log \tfrac{1}{2}[\phi . \{f(x)\} + f(x) + f . \{f(x)\}],$$

where $e$ is the base of the Napierian system of logarithms.

57. Two toothed wheels work against each other. Shew that, if the number of the teeth in one be prime to that in the other, before two teeth which have once been in contact come into contact again, every tooth of the one wheel will have been in contact with every tooth of the other.

Find also the least number of revolutions of one of the wheels when there is one pair of teeth intervening between the two first in contact.

58. A bag contains black and white balls. If one ball be drawn and laid aside and then a second drawn, the chance of the second being white is the same as the chance of the first being so.

59. If three dice be thrown, what is the chance that they will all turn up aces?

60. In a ballot-box there are found 5 white balls, and 2 black ones. What is the probability (1) that any specified person gave a black ball; (2) that a black ball was given by one of two specified persons?

61. A boat's crew consists of eight men, three of whom can row only on one side, and two only on the other. Find the number of ways in which the crew can be arranged.

62. Six cards are to be dealt, viz., three diamonds and three clubs, two of a suit must never be dealt together, shew that there are 72 ways in which they can be dealt away. If an observer watched the first four cards, what odds ought he to give or take that the remaining cards were a diamond and club?

63. $n$ dice, each having $m$ faces, are thrown altogether; shew that the number of different throws which can be made is

$$\frac{m(m+1)\ldots(m+n-1)}{1.2.3\ldots n}.$$

64. There are $n$ lines in a plane, no two of which are parallel, and no three pass through the same point.

Their points of intersection are joined. Shew that the number of fresh lines thus introduced is

$$\tfrac{1}{8}\,n.(n-1).(n-2).(n-3).$$

65. Two boats $A$ and $B$ start for a race against stream. Their velocities through the water are $v$ and $V$. The velocity of the river at its sides is $r$ and in midstream $nr$. $A$ is next the bank at starting, and $B$ in the middle of the river, but at a distance $x$ from the starting-place there is a bend which puts $B$ next the (other) bank and $A$ in midstream. Suppose the length of the course to be $a$; then if $A$ wins we must have

$$x > \frac{a}{n-1} \cdot \frac{\{V - v + (n-1)r\}(v-r)(V-nr)}{r(v^2+V^2) - (n+1)n^2(v+V) + 2nr^3}.$$

66. Assuming that if $a$ be prime to $b$, each of the quantities $a$, $2a$, $3a$, $(b-1)a$ when divided by $b$, will leave a different positive remainder; shew that if $r$ be the first of these remainders and $q$ the least integer which makes $qr$ greater than $b$, then, provided $(q+1)r$ is not greater than $2b$, the above-named remainders will occur in a series of *increasing* arithmetical progressions whose common difference is $r$. But if $(q+1)r$ is greater than $2b$, then the remainders will occur in a series of *decreasing* arithmetical progressions whose common difference is $b-r$.

# THEORY OF EQUATIONS.

1. FORM the rational equation two of whose roots are $2+\sqrt{3}$, $-1-\sqrt{3}$.

2. (1) $3+\sqrt{(-1)}$ is a root of the equation

$$x^4-6x^3+13x^2-18x+30=0,$$

(2) $\sqrt{3}-1$ is a root of the equation $x^4+2x^3-5x^2-6x+6=0$,

(3) $1+\sqrt{2}$ is a root of the equation $x^4-7x^3+15x^2-7x-6=0$,

find all the other roots.

3. Express in terms of the coefficients of an equation the function of its roots $a$, $b$, ...

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{a^2}{c}+\frac{c^2}{a}+\frac{b^2}{c}+\frac{c^2}{b}+\dots$$

4. Shew that the sum of the cubes of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\dots=0$$

is

$$3p_1p_2-3p_3-p_1^3.$$

5. If the roots of the equation

$$x^n-p_1x^{n-1}+p_2x^{n-2}-\dots+(-1)^np_n=0$$

be real, and $p_1, p_2, \dots$ be positive, shew that

$$p_1 . p_{n-1} > \tfrac{1}{2}n . (n+1) p_n.$$

6. Diminish by unity the roots of the equation

$$2x^4 - 17x^3 + 53x^2 - 72x + 36 = 0,$$

and thence solve it.

7. Transform to an equation in which there are no fractional coefficients

$$x^3 - \frac{2}{45}x + \frac{7}{81.625} = 0.$$

8. Solve the equations:

(1) $x^4 - \frac{5}{6}x^3 - \frac{19}{3}x^2 - \frac{5}{6}x + 1 = 0.$

(2) $6x^6 - 23x^5 + 10x^4 + 14x^3 + 10x^2 - 23x + 6 = 0.$

9. If $a$, $b$, $c$, be the roots of the equation $x^3 + qx + r = 0$, form the equation whose roots are

$$ab + \frac{1}{a+b}, \quad bc + \frac{1}{b+c}, \quad ac + \frac{1}{a+c}.$$

10. Find the superior and inferior limits of the positive roots of the equation

$$x^7 + 6x^6 - 51x^4 - 10x^3 - 103x + 500 = 0,$$

and shew that there cannot be more than five real roots.

11. Find superior and inferior limits of all the roots of the equations

(1) $x^6 + 6x^5 - 7x^4 + 2x^3 - 8x - 10 = 0.$

(2) $x^6 + 7x^5 - 103x^4 - 5x^3 - 156x - 5 = 0.$

12. Illustrate geometrically the property that an odd number of roots of $f'(x) = 0$ lie between each adjacent two of $f(x) = 0$, and apply the property to shew when $x^3 - qx + r = 0$ has 3 real roots.

Find the position of the roots of

$$x^3 - 12x + 15 = 0.$$

13. Find the number of real roots of the equation

$$3x^4 - 4(p+q)x^3 + 6pqx^2 - p^2q^2 = 0.$$

14. Illustrate Des Cartes' rule of signs, by discussing the curve whose equation is

$$x^2(y^2 - ax) + a^2y^3 = 0.$$

15. The equation

$$x^4 - 4x^3 + 16x - 16 = 0.$$

has equal roots; find all the roots.

16. Find by Newton's method of divisors the integral roots of the equations,

$$\begin{aligned} &(1)\quad x^4 - 4x^3 - 20x^2 + 95x - 60 = 0, \\ &(2)\quad x^4 + 8x^3 - 7x^2 - 49x + 56 = 0, \\ &(3)\quad x^5 - 5x^4 + 3x^2 + 18x + 16 = 0. \end{aligned}$$

17. Proceed according to Des Cartes' method to find the roots of

$$x^4 - 6x^2 - 8x - 3 = 0.$$

18. Apply Sturm's method to discover the position of the roots of

$$\begin{aligned} &(1)\qquad x^3 - 10x + 4 = 0. \\ &(2)\quad x^4 - 4x^3 - 3x + 23 = 0. \end{aligned}$$

19. Shew that if $f(x) = 0$ has two roots equal and of opposite signs, and $\phi(x)$ contain the even powers, $x\psi(x)$ the odd powers, $\phi(x)$ and $\psi(x)$ have a common divisor, which is the factor containing the roots.

20. Prove the following properties of the curve whose equation is

$$x^4 - 3axy^2 + 2ay^3 = 0,$$

by the theory of equations, $Oy$ measured upwards, $Ox$ to the right:

(1) That all straight lines parallel to the axis of $x$ and above it cut the curve in two points on the right of the axis of $y$, or in none and those below it cut the curve in two points, one on each side of the axis of $y$.

(2) That all straight lines parallel to the axis of $y$ and to the right of it cut the curve in three points, one below and two above the axis of $x$, or in one below it; and those to the left of it cut the curve in one point below the axis of $x$.

(3) That the curve extends in the angle $xOy$ to distances $a$ and $\frac{3^7}{2^{11}}a$ from $Oy$ and $Ox$ respectively.

Trace the curve.

# TRIGONOMETRY.

1. SHEW that there are the same number of grades in the angle of an octagon, as of degrees in that of a dodecagon.

2. A well is 100 feet deep. How many coils of rope will be required to reach the bottom, the roller on which they are wrapped being 8 inches in diameter?

3. A certain coin is $\frac{1}{16}$th of an inch thick and $\frac{2}{3}$ of an inch in diameter. Another has to be made of $2\frac{1}{2}$ times the value and twice as thick. What will be its size? Shew that a cubic inch of metal will make $18\frac{1}{3}$ such coins nearly.

4. Prove that

(1) $$\frac{\cot A \cos A}{\cot A + \cos A} = \frac{\cot A - \cos A}{\cot A \cos A}.$$

(2) $(\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 = (1 + \sec A . \operatorname{cosec} A)^2.$

(3) $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A.$

(4) $\sin(\alpha + 2\theta)\sin\alpha - \sin(\theta + 2\alpha)\sin\theta = \sin^2\alpha - \sin^2\theta.$

5. If $\cot A = 2\tan B$, then $\cos(A - B) = 3\cos(A + B)$.

6. Shew that if

$\cos(A - B) = 3\cos(A + B)$, then $\sec(A + B) = 2\sec A \sec B$.

7. If $\sin 5A - \sin 3A = \tan 2A - \tan A$, prove that

$$\sin 8A = 4\sin A.$$

8. Find the values of $A$ which satisfy the equation

$$\tan 2A = 3\tan A.$$

If $\sin(\theta - \alpha) = \sin\theta - \sin\alpha$, prove that $\theta = 2m\pi$, or $2m\pi + \alpha$.

9. Find all the solutions of

$$1-\cos A=2\sin^2 A,$$

and reduce them under one formula.

10. Find all the values of $\theta$ which satisfy the equations

(1) $\cos\theta+\cos 3\theta+\cos 5\theta=0$.

(2) $\sin\theta=\tan\theta-\tan 2\theta$.

(3) $\cot\theta-\tan\theta=\cos\theta+\sin\theta$.

(4) $\sin 5\theta-\sin 3\theta=\cos 6\theta+\cos 2\theta$.

(5) $\cos 6\theta-\cos 2\theta=\sin 2\theta\sec 4\theta$.

(6) $\cos 8\theta-\cos 5\theta+\cos 3\theta=1$.

(7) $\sin\theta+\sin 2\theta+\sin 3\theta=1+\cos\theta+\cos 2\theta$.

(8) $\tan\theta\tan n\theta=1$.

11. Solve the equation

$$\tan(\cot x)=\cot(\tan x).$$

12. Find the values of $\theta$ and $\tan\theta$ which satisfy the condition that $\tan\theta=\tan 4\theta$, and find the values of $\theta$ when $2\sin n\theta=1$.

13. Find $\tan x$ from the equation

$$3\tan x\tan 3x+1=0.$$

14. Find $x$ from the equation

$$\cos^{-1}\frac{1-x^2}{1+x^2}+\tan^{-1}\frac{2x}{1-x^2}=\frac{4\pi}{3}.$$

15. Find $x$ and $y$ from the equations

$$\sin x=\tan\theta\sin(y+\theta),$$
$$\sin(x-\theta)=\cos y-\cos(y+2\theta)-1.$$

16. Prove that

(1) $\tan(A-B).\tan(B-C).\tan(C-A)=\tan(A-B)+\tan(B-C)+\tan(C-A)$.

(2) $\tan^{-1}(\cos 2\theta) = \frac{1}{2}.\tan^{-1}\{\frac{1}{2}(\cot^2\theta - \tan^2\theta)\}$,

and shew that if $A+B+C=180$, then will

$$\sin^2 A + \sin^2 B + \sin^2 C = 2(\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B).$$

17. Eliminate $\phi$ and $\theta$ from the equations

$$a\cos\phi + b\cos\theta = c, \qquad a\sin\phi - b\sin\theta = d, \qquad 2\sin^2\frac{\phi+\theta}{2} = \cos\frac{\pi}{3}.$$

18. If $\sin A.\sin B = \sin(P+Q).\sin R$,

$\cos A.\cos B = \cos(P+Q).\cos R$,

and $(\cos A)^2 + (\cos B)^2 - \{\cos(P+Q+R)\}^2 = 1$;

then $\{\sin(P+Q)\}^2 + (\sin R)^2 = \{\sin(P+Q+R)\}^2$.

19. Having given that

$$1 + \cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = 0,$$

shew that $\alpha \backsim \beta$, $\beta \backsim \gamma$, or $\gamma \backsim \alpha$ is equal to an odd multiple of $\pi$.

20. If $n\sqrt{2}.\sin\frac{\phi}{2} = \sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$ and $\sin\phi = \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$,

$$\text{then } \tan\beta = \left(\frac{1}{n}-1\right)^2 \tan\alpha.$$

21. If $1+\cos\phi\cos\psi = \sin\phi\sin\psi\,\text{cosec}\,\theta$, then will each member of the equation $= \pm(\cos\phi+\cos\psi)\sec\theta$.

22. Eliminate $\phi$ from the equations

$$x\cos(\phi+\alpha) + y\sin(\phi+\alpha) = a\sin 2\phi,$$
$$y\cos(\phi+\alpha) - x\sin(\phi+\alpha) = 2a\cos 2\phi,$$

shewing that

$$(x\sin\alpha - y\cos\alpha)^{\frac{2}{3}} + (y\sin\alpha + x\cos\alpha)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

23. $AB$, $BD$ are two radii of a circle at right angles to each other. Produce $BD$ to $C$, and make $BC$ equal to the arc $AD$. Join $AC$ cutting the circumference in $E$. Then the area $EDC$ = area of the segment $AE$.

24. If the arcs of two circles radii $r$, $r'$ cut each other at an angle $\theta$, then the included area

$$= r^2\theta - (r'^2 - r^2)\tan^{-1}\frac{r\sin\theta}{r' + r\cos\theta} - rr'\sin\theta.$$

25. If $d$ be the perpendicular dropt from the angle $C$ on the opposite side of a triangle, shew that $\sin C = \dfrac{c.d}{a.b}$.

Deduce the formula

$$\sin(A+B) = \sin A.\cos B + \cos A.\sin B.$$

26. Shew that in a triangle $ABC$ if $C$ be a right angle,

$$(1)\quad abc = a^3.\cos A + b^3.\cos B,$$

$$(2)\quad \tan^{-1}\frac{a}{b+c} + \tan^{-1}\frac{b}{a+c} = \frac{\pi}{4}.$$

27. Prove that in any triangle

$$(1)\quad \cot B - \cot A = \frac{a^2-b^2}{ab}\,\frac{1}{\sin C}.$$

$$(2)\quad \frac{\sin\left(\dfrac{A}{2}+B\right)}{\sin\dfrac{A}{2}} = \frac{b+c}{a}.$$

$$(3)\quad \text{the area} = \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\left(\frac{a^2}{\sin A} + \frac{b^2}{\sin B} + \frac{c^2}{\sin C}\right).$$

28. A pole is fixed on the top of a mound and the angles of elevation of the top and bottom of the pole are $60^\circ$ and $30^\circ$; shew that the length of the pole is twice the height of the mound.

29. A person at a distance ($a$) from a tower which stands on a horizontal plane, observes that the angle of elevation ($\alpha$) of its highest point is the complement of that of a flag-staff on the top of it. Shew that the length of the flag-staff is

$$2a \,.\, \cot 2\alpha.$$

If the distance of the person from the tower is unknown, and if, when he recedes a distance ($c$), the angle of elevation of the tower is half of what it was before, shew that the length of the flag-staff is

$$c \,.\, \operatorname{cosec} \alpha \,.\, \cos 2\alpha.$$

30. Two spectators, at two given stations, observe, at the same time, the altitude of a kite, and find it to subtend the same angle $\alpha$ at each place. The angle which the line joining one station and the kite subtends at the other station is $\beta$, and the distance between the two stations is known; find the height of the kite.

31. Two towers stand on a horizontal plane, and their distance is 120 feet. A person standing successively at their bases observes that the angular elevation of one is double of that of the other; but when he is half-way between them their elevations appear complementary to each other. Shew that the heights of the towers are 90 and 40 feet respectively.

32. A person observes the angular elevation of a balloon due east of him to be 45°. He then travels in a north-easterly direction, while the balloon moves southward until it is due south of him, when its angular elevation appears half what it was before. Shew that he and the balloon have passed over equal distances.

33. A church consists of a nave having a tower capped with a spire as its western extremity. The length of the nave is 100 feet and the tower stands on a square base of 30 feet. At a distance of 85 feet from the east end, the top of the spire can be just seen over the gable of the roof at an elevation of $\tan^{-1}\frac{1}{2}$, while at a distance of $22\frac{1}{2}$ ft. to the west of the tower the observer finds himself in the plane of a face of the spire.

Obtain the heights of the tower, spire, and gable.

34. If in the "ambiguous case" $c$, $c'$ be the two values of the third side, and the given angle be $45^\circ$, shew that the angle between the two positions of the side opposite is

$$\cos^{-1}\frac{2cc'}{c^2+c'^2}.$$

35. Find the angle of the sector in which the chord of the arc is three times the radius of the circle inscribed in the sector.

36. Shew that the square described round a circle is equal to $\frac{4}{3}$ of the inscribed duodecagon.

37. If the side $AB$ of an equilateral triangle $ABC$ be produced to $D$, so that $BD=AB$, and $CD$ be joined, shew that the radii of the circles circumscribed about and inscribed in the triangle $BCD$ are $AB$ and $AB\left(\sqrt{3}-\frac{3}{2}\right)$.

38. Shew that if two triangles whose semiperimeters are $S$, $S'$ have equal escribed circles on the sides $a$, $a'$, then the ratio of the radii of their inscribed circles is as

$$\frac{S-a}{S} : \frac{S'-a'}{S'}.$$

39. $ABC$ is a triangle, $BD$ bisects the angle $B$, $CD$ bisects the side $AB$, prove that

$$\frac{\triangle DAB}{c}=\frac{\triangle DBC}{a}=\frac{\triangle DCA}{a}=\frac{\triangle ABC}{2a+c}.$$

40. In the hypothenuse $AB$ of a right-angled triangle $ABC$ two points $D$, $E$, are taken, such that $AE=AC$, $BD=BC$.

Shew that if $CE$ and $CD$ are joined,

$$CD^2 : CE^2 :: b+c : a+c.$$

41. $A$, $B$ are two inaccessible points on a horizontal plane, and $C$, $D$ are two stations at each of which $AB$ is observed to subtend the angle $30^\circ$, $AD$ subtends at $C$, $19^\circ\,15'$ and $AC$ at $D$, $40^\circ\,45'$, shew that $AB=\frac{CD}{\sqrt{3}}$.

42. Shew that in any triangle $ABC$,

$$a.\sin A+b.\sin B+c.\sin C=2.\{\alpha.\cos A+\beta.\cos B+\gamma.\cos C\},$$

when $\alpha$, $\beta$, $\gamma$, are the perpendiculars dropped from the angles $A$, $B$, $C$, on the opposite sides.

43. In any triangle, if the radii of the circles, touching two sides produced and one externally, be in harmonic progression, the mean radius is three times that of the inscribed circle, and the sides are in arithmetical progression.

44. $ABC$ is a triangle, $Aa$ bisects $BC$ in $a$ and is produced to $\alpha$ so that $A\alpha=\frac{4}{3}Aa$, the same construction is made for $B\beta$, $C\gamma$. Shew that $\alpha\beta\gamma$ is a triangle similar to $ABC$ and equal to it. Shew that the hexagon common to the two triangles is $\frac{2}{3}$ of either triangle.

45. If in a triangle $ABC$, $BC=70$, $AC=35$, and $\angle ACB=36^\circ 52' 12''$, find the remaining angles, the tables giving

$$\log 3=.4771213,$$

$$\log\cos 18^\circ 26' 6''=10.4771213.$$

46. The space in the Menai Bridge, between the two pillars supporting the chain at the highest point, is $a$ feet, and a point on the beach is observed with a sextant from the two extremities of the broadway; $\alpha$ and $\beta$ are the angles subtended successively at each extremity by the point and the other extremity. From the observed point the altitude of the extremity at which $\alpha$ was taken is $\gamma$. Shew that the height of the broadway above the level of the sea is

$$\frac{a.\sin\beta.\sin\gamma}{\sin(\alpha+\beta)}.$$

47. A person stands in the diagonal produced of the square base of St. Mary's Church tower, at a distance ($a$) from it; and observes the angles of elevation of the two outer corners of the top of the tower to be each $30^\circ$, and of the other $45^\circ$. Shew that the breadth of the tower is

$$a\sqrt{(3-\sqrt{5})}.$$

48. $D$ is the middle point of the side $BC$ of the plane triangle $ABC$; shew that twice the rectangle contained under $AD$ and $BC$ is equal to

$$(b^4+c^4-2b^2c^2 . \cos 2A)^{\frac{1}{2}}.$$

49. If $a$, $b$, $c$, the sides of a triangle, be in harmonic progression, shew that

$$\left\{\frac{\sin\frac{A}{2}}{\sin\frac{C}{2}}\right\}^2=\frac{\cos B-\cos A}{\cos C-\cos B}.$$

50. If $A$, $B$, $C$ be the angles of a triangle,

$$l^2+m^2+n^2>2mn\cos A+2ln\cos B+2lm\cos C,$$

unless $$l:m:n::\sin A:\sin B:\sin C.$$

51. From the angles $A$ and $B$ of a triangle $ABC$, lines are drawn bisecting the opposite sides in $D$ and $E$; find a point in $AB$ at which $AE$ and $BD$ subtend equal angles, and shew that the cotangent of either of these angles is in arithmetic mean between $\cot A$ and $\cot B$.

52. $AB$ the diameter of a circle is produced both ways to $C$, $D$, so that $CA=AB=BD$. If $P$ be any point in the circle, shew that

$$\tan CPA . \tan BPD=\tfrac{1}{4}.$$

53. Perpendiculars are drawn from the angles of a triangle on the opposite sides, and produced to the circumscribed circle in $A'B'C'$, prove that

$$\frac{\cos(B-C)}{AA'}=\frac{\cos(C-A)}{BB'}=\frac{\cos(A-B)}{CC'}=\frac{2ABC}{abc}.$$

54. Let $ABCD$ be a quadrilateral figure inscribed in a circle, $d_1$, $d_2$ its diagonals intersecting in $E$, $r_1$, $r_2$, $r_3$, $r_4$, the radii of the circles described about the triangles $ABE$, $BCE$, $CDE$, $DAE$. If the angle between the diagonals be one-third of a right angle, shew that

$$r_1r_3+r_2r_4=d_1d_2.$$

55. $ABC$ is a triangle right-angled at $A$; from $A$ draw $AD$ perpendicular to $BC$; shew that if $r_1$, $r_2$ be the radii of the circles which touch $CB$, $DA$ produced, and $AB$, and $DA$, $BC$ produced, and $AC$, and $r$ the radius of the circle which touches $AB$, $AC$ produced, and $BC$, then

$$r_1^2 + r_2^2 = r^2.$$

56. Through the exterior angles $A$, $B$, $C$ of a triangle, lines are drawn inclined at the same angle successively to the sides $c$, $a$, $b$. If $R_1$, $R_2$, $R_3$, be the radii of circles circumscribing the three triangles which are cut out of the triangle so formed by the triangle $ABC$, $R$ that of the circle circumscribing $ABC$; shew that

$$R_1R_2R_3 = R^3.$$

57. $A_1$, $A_2$, $A_3$, are the areas of triangles having respectively the sides $a$, $b$, $c$, for bases and the centre of the inscribed circle for vertex. $A_1'$, $A_2'$, $A_3'$, those of triangles having the same bases and the centres of the respective escribed circles for vertices; shew that

$$\frac{A_1}{A_1'} + \frac{A_2}{A_2'} + \frac{A_3}{A_3'} = 1,$$

$$\frac{a}{A_1'} + \frac{b}{A_2'} + \frac{c}{A_3'} = \frac{2}{r},$$

$r$ being the radius of the inscribed circle.

58. Three circles, whose radii are $r_1$, $r_2$, $r_3$, touch one another externally. Shew that one of the sides of the triangle made by joining the points of contact, is

$$2r_1\left\{\frac{r_2r_3}{(r_1+r_2)(r_1+r_3)}\right\}^{\frac{1}{2}};$$

and its area : the area of the triangle formed by joining the centres of the circles as

$$2(r_1r_2r_3) : (r_1+r_2)(r_1+r_3)(r_2+r_3).$$

59. If $ABC$ be a triangle, and tangents be drawn to the circle inscribed in it parallel to the sides $BC$, $AB$, $AC$, respectively, and $r_a$, $r_b$, $r_c$ be the radii of the circles inscribed in the triangles formed by these tangents and the sides of the

triangle $ABC$, and $r$ the radius of the circle inscribed in the triangle $ABC$, then

$$r_a + r_b + r_c = r.$$

60. An observer, standing outside a square enclosure, sees the two sides nearest to him under the angles $\alpha$ and $\beta$. If these sides face the south and west, shew that his position is $\theta$ degrees west of south, where

$$\tan\theta = \frac{1+\cot\alpha}{1+\cot\beta}.$$

61. A tower standing on a horizontal plane is surrounded by a moat which is just as wide as the tower is high. A person on the top of another tower whose height is $(a)$, and whose distance from the moat is $(c)$, observes that the first tower subtends an angle of 45°. Shew that the height of the first tower is

$$\frac{a^2+c^2}{a-c}.$$

62. $ABC$ is a triangle right angled at $C$ and having the angle $BAC = \frac{\pi}{n}$. On $AB$ is described another triangle $ABB'$ right angled at $B$ and having the angle $BAB = \frac{\pi}{n}$, and so on. If $A$ be the area of the triangle $ABC$, and $\Sigma(A)$ the sum of all the triangles which exactly fill up the angular space at $A$, shew that

$$\frac{\Sigma(A)}{A} = \cot^2\frac{\pi}{n}\left(\sec^{4n}\frac{\pi}{n} - 1\right).$$

63. A circle is drawn to touch the circumscribing circle of a triangle $ABC$, and the two sides $AB$, $AC$ produced: if $r$ be its radius, and $R$ the radius of the circumscribing circle, shew that

$$r\cos^2\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

If $B = C$, find the value of $A$ in order that $r$ may be equal to $R$.

64. The altitude of a mountain ($\alpha$) is taken at a station $A$, the observer then recedes through a space $c$, which he considers level, and observes the angle $\beta$ between $A$ and the top of the mountain; shew that he obtains the height

$$c \sin \alpha \sin \beta \operatorname{cosec} (\alpha - \beta).$$

If his imagined level was in reality a slight rise of $\theta$, shew that his error has increased the height by a quantity bearing to the height the ratio of

$$\theta : \tan (\alpha - \beta).$$

65. If two roofs whose ridges are at right angles to each other make angles $\alpha$ and $\beta$ with the horizon, then the angle ($\theta$) which their intersection makes with it is given by the equation

$$\cot^2 \theta = \cot^2 \alpha + \cot^2 \beta.$$

66. A triangle is described on a given base between parallels whose distance equals half that base. Shew that the ratio of the sides

$$= \sqrt{(1 + \sin 2A)} + \sqrt{(\sin 2A)},$$

$A$ being the vertical angle.

67. The diameter and three adjacent sides of a regular hexagon are supposed to be connected by hinges. Shew that if the two extreme sides be placed so as to make angles $\theta$, $\phi$, with the diameter, then

$$4 (\cos \theta + \cos \phi) - 2 \cos (\theta + \phi) = 5.$$

68. Find the area common to two equilateral triangles inscribed in a circle, the position of one differing by a given angle from that of the other, and shew that when the area in question becomes a regular hexagon it will be equal to

$$r^2 \frac{\sqrt{3}}{2}.$$

69. $ABC$ is a triangle; $cAb$ is drawn through $A$ making angles $B$, $C$, with $AC$, $BC$ respectively, $cBa$ through $B$, $aCb$ through $C$, so that the angle $aBC = A = aCB$. Shew that the area of the triangle $abc$ : area of triangle $ABC$ as

$$1 : 2 \cos A . \cos B . \cos C.$$

70. A point $O$ is taken within a triangle $ABC$ such that

$$\triangle OAB : \triangle OBC : \triangle OCA :: p : q : r;$$

find $OA^2$, $OB^2$, and $OC^2$, and shew that if the triangle is equilateral

$$(p-r)\,OA^2+(q-p)\,OB^2+(r-q)\,OC^2=0.$$

71. A person wishing to ascertain the height of two trees standing on a horizontal plane, places himself in the same straight line with them and observes that the angle between their summits is $\alpha°$. He then walks towards them till their summits coincide, when he finds that their common angle of elevation is equal to twice that of the more distant summit at the first station. He again walks on till at a certain point between them he finds their angles of elevation to be each equal to $\beta°$. If $\theta$ be the elevation of the more distant summit at the first station, shew that

$$\cot\theta=(1+2\cot\alpha.\cot\beta)^{\frac{1}{2}},$$

and deduce the heights of the two trees.

72. If the centers of the escribed circles of a triangle be joined so as to form another triangle, then the lines joining those centers with the points in which the circles touch the original triangle will pass through the center of the circle described about the second triangle.

73. If $A$ be the area of a triangle, $A_1$ the area of the triangle formed by joining the points where the escribed circles touch the triangle, $A_2$ the area of the triangle formed by joining the centres of the escribed circles, then

$$A_1 : A :: A : A_2.$$

74. If a triangle be formed by joining the points of contact of the inscribed circle, and a second circle, radius $r'$, be inscribed in it, then

$$\frac{r'}{r}=\frac{1}{2}\,\frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\left(45°-\frac{A}{4}\right).\cos\left(45°-\frac{B}{4}\right).\cos\left(45°-\frac{C}{4}\right)}.$$

75. If $s$ be the sum of two adjacent sides of a rectangle, $d$ its diagonal, and another rectangle be formed within it by drawing lines at a distance $c$ from its sides, then the angle at which the diagonals of the two rectangles are inclined to one another is given by the equation

$$\tan\theta = \frac{2c\sqrt{(2d^2 - s^2)}}{d^2 - 2cs}.$$

76. A fork shaped like a Y is placed perpendicularly across the edge of a prism with its branches equally inclined to each face. The angle of the prism $(2\alpha)$ being less than that of the fork $(2\beta)$, shew that the angle $(\theta)$ through which it must be turned about its stem so as to bring its branches into contact with the sides of the prism will be given by the equation

$$\cos\theta = \tan\alpha \cot\beta.$$

77. Three candles are placed on a table at the angles of a known triangle. If the heights of the candles be given, find the lengths of the shadows of the two shorter ones, and prove that if the heights of the candles be in any way altered so that the lengths of two of the shadows remain the same, that of the third will also remain the same.

78. A tower stands at the foot of a hill, whose side makes an angle $\alpha$ with the horizontal plane on which the tower stands, and it is observed that when the Sun's altitude is $\beta$, the extremity of the shadow falls at a point on the hill's side, whose height above the horizontal plane is $c$. If the direction of the hill's side makes an angle $\gamma$ with the vertical plane passing through the Sun and the tower, shew that the tower's height will exceed that of the point by

$$c \,.\, \cot\alpha \,.\, \tan\beta \,.\, \operatorname{cosec}\gamma.$$

79. Two cones of different vertical angles are placed one within the other and the inner one made to roll round and round. Shew that the same generating lines will never be twice in contact unless the sines of the semivertical angles of the cones have the same surd factor.

80. Each of the $2r$ sides of a regular polygon is divided in the ratio of $m : n$, the adjacent segments in consecutive sides being equal. If the points of section be joined, so as to form another polygon of $2r$ sides, shew that the area of this polygon will be to that of the original polygon as

$$1 - 2r\frac{m^2+n^2}{(m+n)^2} \cdot \sin^2\frac{\pi}{2r} : 1.$$

81. A pyramid, whose faces are equilateral triangles and base a square, stands on a horizontal plane, and faces the cardinal points.

The Sun's altitude being $\alpha$, and $\beta$ his distance from the meridian, shew that $\theta$, the vertical angle of the shadow, is given by the equation

$$\tan\theta = \tan 2\alpha \cos\left(\frac{\pi}{4} - \beta\right).$$

82. Shew that the real part of

$$\log(1 + \sqrt{-1}\tan\alpha) \text{ is } \log(\sec\alpha).$$

Find the real part of $\tan^{-1}(\alpha + \beta\sqrt{-1})$.

83. Find $A$ when $\tan A = 1.01$.

84. If $\alpha$, $\beta$ be the roots of $x^2 + x + 1 = 0$, shew that

$$\frac{1}{1+x+x^2} = \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right);$$

and thence shew that the coefficient of $x^n$ in the expansion of

$$\frac{1}{1+x+x^2} \text{ is } \sin\frac{2(n+1)\pi}{3} \div \sin\frac{2\pi}{3}.$$

85. In the expansion of $\dfrac{\cos\theta - x}{x^2 - 2x\cos\theta + 1}$ the coefficient of $x^n$ is $\cos(n+1)\theta$.

86. Prove that if $n$ be a positive integer, the real part of

$$\frac{\{\sqrt{(-1)}\}^n - 1}{\sqrt{(-1)} - 1} = \frac{1}{2}\left(1 + \sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right).$$

87. If $2\tan\alpha = 3\alpha$, then $\alpha = \frac{\pi}{3} - \frac{1}{5\pi}$, nearly.

88. Prove that

$$\log_\epsilon\left(2^{\frac{1}{2}}.\cos\frac{\theta}{2}\right) = \frac{\cos\theta}{2} - \frac{\cos^2\theta}{4} + \frac{\cos^3\theta}{6} - \ldots,$$

and find the sum of $n$ terms of the series

$$\sin^2\theta.\sin 2\theta + \tfrac{1}{2}.\sin^2 2\theta.\sin 2^2\theta + \frac{1}{2^2}.\sin^2 2^2\theta.\sin 2^3\theta + \ldots$$

89. Find the sum of $n$ terms of the series

$$\cos 2\theta \operatorname{cosec}\theta + \cos 2^2\theta \operatorname{cosec} 2\theta + \cos 2^3\theta \operatorname{cosec} 4\theta + \ldots \&c.$$

90. Sum $\tan\theta\sec 2\theta + \tan 2\theta \sec 4\theta + \tan 4\theta \sec 8\theta + \ldots$ to $n$ terms, and shew that

$$\cos\theta - \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta \ldots \text{ to } \infty = \frac{\pi}{4}.$$

91. Prove that the sum $n+1$ terms of the series

$$\sin\alpha + n\sin 2\alpha + n.\frac{n-1}{2}\sin 3\alpha + \ldots\ldots \text{ is } 2^n \sin\frac{n+2}{2}\alpha\cos^n\frac{\alpha}{2}.$$

92. Shew that

$$\cos\theta - \sin\theta + \tfrac{1}{3}(\cos^3\theta - \sin^3\theta) + \tfrac{1}{5}(\cos^5\theta - \sin^5\theta) + \ldots$$
$$= \log\left\{\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}.$$

93. If $\sin\frac{\theta}{2} = m\sin\left(2\alpha - \frac{\theta}{2}\right)$,

prove that $\theta = 2m\sin 2\alpha - m^2\sin 4\alpha + 2\frac{m^2}{3}\sin 6\alpha - \&c.$

94. If $m\sin(m\phi + \theta) = \sin m\phi$, then

$$\phi = \sin\theta + \tfrac{1}{2}m\sin 2\theta + \tfrac{1}{3}m^2\sin 3\theta + \&c.$$

95. A string is partially wrapped upon two cylinders whose radii are $a$ and $b$. The portion not wrapped on either is kept stretched, its initial length being $l$.

One cylinder being carried round the other through an angle $\theta$, and always kept in such a position that any radius continues parallel to itself, shew that when the cylinders come into contact

$$\theta = \tfrac{1}{3}\frac{l^3}{(a+b)^3} - \tfrac{1}{5}\frac{l^5}{(a+b)^5} + \dots$$

N.B. The string wraps on both.

96. Shew by the equality of the coefficients of $x^n$ in the expansions of

$$e^x \sin x \times e^x \cos x, \text{ and of } \frac{e^{2x}\sin 2x}{2},$$

or otherwise, that

$$\Sigma_{r=0}^{r=n-1} \frac{\sin (n-r)\frac{\pi}{4}\cos\frac{r\pi}{4}}{\lfloor r \lfloor n-r} = \frac{2^{n-1}\sin\frac{n\pi}{4}}{\lfloor n}.$$

# ANALYTICAL GEOMETRY OF TWO DIMENSIONS.

1. Determine the loci represented by the equations

(1) $x^3 - 4a^2x = 0.$

(2) $x^2 - 2ax - 3a^2 = 0.$

(3) $x^4 - y^4 = 0.$

(4) $xy - 2ay + 3ax - 6a^2 = 0.$

(5) $x^2 - y^2 - 3ax + 3ay = 0.$

(6) $25x^2 - 9y^2 - 10x + 6y = 0.$

2. Find the co-ordinates of the points which bisect the line joining the points whose co-ordinates are

$$(2a, -b), \quad (-a, 2b),$$

and shew that the area of the triangle included between the straight lines whose equations are

$$x + 2y = 2a, \quad x + y = 2a, \quad \text{and } x = 0 \text{ is } a^2.$$

3. Find the points of intersection of the straight lines whose equations are

$$x + 2y - 5 = 0, \quad 2x + y - 7 = 0, \quad \text{and } y - x - 1 = 0;$$

and shew that the area of the triangle formed by them is $\frac{3}{2}$.

4. Shew that if the whole area included between the lines

$$x + y = c,$$

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and the co-ordinate axes, be bisected by the line which joins

their point of intersection with the origin, then $c$ is a geometric mean between $a$ and $b$.

5. Find the length of the perpendicular from the point $(a, 0)$ on the straight line whose equation is

$$\frac{x}{a}+\frac{y}{b}=\frac{c}{\sqrt{(a^2+b^2)}}.$$

6. The equations of two straight lines $APB$ and $CPD$ are

$$x+3y-a=0, \quad \text{and} \quad y-x+a=0.$$

Find the equations of the two straight lines passing through $P$ their point of intersection, such that the ratio of the sines of inclination to $AB$ and $CD$ may be as $1 : \sqrt{5}$.

7. Find the equations of the lines bisecting the angles between the lines whose equations are

$$12x+5y=8, \quad \text{and} \quad 3x-4y=3.$$

8. Find the equation of the straight line which passes through the point of intersection of the straight lines whose equations are

$$x-2y-a=0, \quad \text{and} \quad x+3y-2a=0,$$

and is parallel to that whose equation is

$$3x+4y=0.$$

9. Find the equation of the straight line which joins the points of intersection of the two pairs of straight lines whose equations are

$$\left.\begin{array}{r}2x+3y-4a=0\\2x+\ y-\ a=0\end{array}\right\} \quad \text{and} \quad \left.\begin{array}{r}x+6y-7a=0\\3x-2y+2a=0\end{array}\right\}.$$

10. Find the condition of parallelism and perpendicularity of the straight lines represented by the equations

$$x+(a+b)y+c=0,$$
$$a(x+ay)+b(x-by)+d=0.$$

11. Determine the loci represented by the polar equations

$$\theta-\alpha=0 \ (1), \quad \sin(\theta-\alpha)=0 \ (2),$$
$$\text{and} \quad (r-a)\,(r^2-a^2-ar\tan\theta\sin\theta)=0 \ (3).$$

12. The equation of a curve is $x^2 + \frac{5}{2}xy + y^2 = a^2$; find the equation when referred to axes inclined at 45° to the given rectangular axes.

13. If $\alpha = 0, \quad \beta = 0, \quad \gamma = 0,$
be the equations to the sides of a triangle, shew that the equation to the line joining $A$ with the point in which the line bisecting $C$ meets a line through $B$ parallel to $CA$ is

$$\beta \sin A + \gamma \sin C = 0.$$

14. If $\alpha = 0, \quad \beta = 0, \quad \gamma = 0,$
be the equations to the sides of a triangle, shew that the equation to the line which joins the middle point of $AB$ with the point of intersection of the lines which bisect the angles $BAC$, $ABC$, is

$$a\alpha - b\beta - (a - b)\gamma = 0.$$

15. If $\alpha = 0, \quad \beta = 0,$
be the equations of two straight lines, and a parallelogram be formed by drawing straight lines parallel to these at distances $a$, $b$, respectively, find the equations of the diagonals.

16. Let $ABC$ be a triangle and let $P$, $Q$ divide $BC$, and $AC$ in the ratio $n : m$, find the equations of $AP$, $BQ$; and if $CR$ passes through the point of intersection, shew that $CR$ bisects $AB$, or divides it in the ratio $m^2 : n^2$ according to the order in which $BC$, $AC$ are divided.

17. If straight lines be drawn bisecting the angles of a polygon of $n$ sides, shew that, if $n-1$ of the bisectors pass through a fixed point, the remaining one will also pass through the same.

18. $Ox$, $Oy$ are two straight lines, $AB$, $A'B'$ are parallel lines drawn from fixed points $A$, $A'$ in $Ox$ and meeting $Oy$ in $B$, $B'$, shew that the locus of the intersection of $AB'$ and $A'B$ is a straight line parallel to $Oy$ which cuts $Ox$ in $P$ so that $OA$, $2OP$, and $OA'$ are in harmonic progression.

19. Find the equations to the straight lines joining the center of the circle $x^2 + y^2 = a^2$, with the points in which the line $2(x + y) = a$ meets it.

20. Find the equations of the two circles which touch the straight lines whose equations are $y \pm x = 0$, and pass through a point $(h, k)$.

21. Find the center of the circle whose equation is

$$x^2 + xy + y^2 - 10x - 8y = 0.$$

22. The diameters $(2a)$, $(2b)$ of two circles make up the diameter of a greater circle. The radius of the greater circle perpendicular to the common diameter meets the circle in $P$. If tangents be drawn from $P$ to touch the smaller circles, shew that they are in the ratio

$$\sqrt{b} : \sqrt{a}.$$

23. If $\frac{x}{a} + \frac{y}{b} = 1$ be the equation to a chord of the circle $x^2 + y^2 = c^2$, shew that the pairs of tangents at its extremities intersect in a point $\frac{c^2}{a}, \frac{c^2}{b}$; and that the tangents of the angles which these tangents make with the axis of $x$ are given by the equation

$$\frac{c^2 - b^2}{b^2} - \frac{2c^2}{ab}\, t + \frac{c^2 - a^2}{a^2}\, t^2 = 0.$$

24. Find the equation of the straight line joining the origin with the point of intersection of the three chords of the circles whose equations are

$$x^2 + y^2 + x + y + 1 = 0,$$
$$x^2 + y^2 + 2x + 2y + 1 = 0,$$
$$x^2 + y^2 + 3x + 3y + 2 = 0.$$

25. Shew that the portion of the line

$$x + y = 3a,$$

which is intercepted between the co-ordinate axes, is trisected at the points where it meets the circle

$$x^2 + y^2 + a^2 = 2a(x + y).$$

Shew also that the parabola $y^2 = 4ax$, passes through one of the points of intersection of the line and circle.

26. Find the locus of a point whose distance from a fixed straight line is equal to its distance from a circle given in position.

27. Find the locus of the middle points of the straight lines touching the circle

$$(x-a)^2+(y-b)^2=c^2,$$

and terminated by the co-ordinate axes to which it is referred. Shew that if the locus be represented by two coincident straight lines the circle passes through the origin.

28. Any number of equal circles intersect a given circle, so that all the chords of intersection pass through a given point. Shew analytically, that the centers of the intersecting circles all lie in a circle whose center is the given point.

29. $ACB$ is the diameter of a circle, $CP$, $CQ$, are perpendicular radii, shew that the locus of the intersection of $AP$, and $BQ$ is a circle whose center is in the given circle, and radius the diagonal of the square on the radius.

30. Find the eccentricity of the ellipse whose equation is

$$x^2+4y^2=9,$$

and the length of the perpendicular on the tangent from the center, which is inclined at an angle $\cot^{-1}(2\sqrt{2})$ to the axis of $x$.

31. In the ellipse whose equation is given in Prop. 30, find the locus of the middle points, (1) of all chords passing through the point $\left(a, \dfrac{2a}{3}\right)$, and (2) of all chords which touch the circle whose equation is $x^2+y^2=a^2$.

32. The equation of an ellipse being $3x^2+4y^2=7$, and of a straight line $x+y=2$, find the equations of the diameters passing through the points of intersection of the ellipse and straight line, and also the co-ordinates of the points where the line meets the directrices.

33. The tangent to an ellipse at the extremity of the latus rectum meets the minor axis produced in $K$, and the normal

at the same point meets the major axis in $G$; shew that if $e$ be the excentricity, the angle which $KG$ makes with the major axis is

$$\cot^{-1} e^3.$$

34. Prove that if $\theta$ be the angle between $SP$ and the tangent at $P\ (x', y')$,

$$\tan\theta = \frac{b^2}{aey'}.$$

35. $AA'$ is the major axis of an ellipse, $P$ any point in the curve; join $AP$, $A'P$, and let $A'P$ meet the tangent at $A$ in $R$; through $R$ draw $RQ$ parallel to $AA'$ meeting $AP$ produced in $Q$; shew that the locus of $Q$ is a parabola, whose latus rectum is equal to that of the ellipse.

Shew also that if $e$ be the excentricity of the ellipse, the distances between the focus of the parabola, and the foci of the ellipse, are to each other as

$$(1-e)^2 : (1+e)^2.$$

36. $a$, $b$ are the semiaxes of an ellipse, and another ellipse has axes in the same lines, and the sum of the axes the same as in the first; shew that the points of alternate intersection are

$$\left\{\pm\sqrt{\left(\frac{a^3}{a+b}\right)}, \text{ and } \pm\sqrt{\left(\frac{b^3}{a+b}\right)}\right\}.$$

Shew that they both touch the curve whose equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a+b)^{\frac{2}{3}}$$

37. $OA$, $OB$ are perpendicular radii of a circle, $OM$, $MP$ co-ordinates of a point $P$ in the circle referred to these axes. A parabola $OQP$ is drawn having its axis in $OA$, and vertex at $O$, and passing through a point $Q$ whose abscissa is $\frac{OM}{2}$; shew that the locus of $Q$ is an ellipse whose semiaxes are $\frac{OA}{2}$ and $\frac{OA}{\sqrt{2}}$ respectively.

38. A circle of radius $b$ touches a parabola of latus rectum $4a$ at the vertex, shew that if it cuts the parabola again, the distance of the straight line joining the points of intersection from the vertex is

$$2b-4a.$$

39. If through $G$ the foot of the normal at any point in a parabola, chords be drawn perpendicular to the normal, prove the following properties: (1) the locus of the middle points of all such chords is a parabola whose latus rectum is one-third of the latus rectum of the given parabola, and vertex at a distance equal to one-half of the latus rectum from its vertex; (2) the locus of the intersection of tangents at the extremities of the chords is an equal parabola turned in the opposite direction, and having the same directrix.

40. Two parabolas have a common axis, and the vertex of one at the focus of the other. If the latus rectum of one be half that of the other, shew that the locus of middle points of all chords of the second, which touch the first, is another parabola, whose latus rectum is an harmonic mean between the given latera recta.

41. If the chord $LP$ be the normal at the extremity $L$ of the latus rectum of $LSl$ in the parabola, shew that the tangents at $L$ and $P$ will intersect in the diameter passing through the other extremity of the latus rectum.

42. $QOQ'$ is a fixed chord in a parabola, $P'OP$ another chord parallel to a given straight line. $Op$ is taken in $OP$ produced if necessary such that

$$OP.Op = OQ.OQ'.$$

Shew that $p$ is a point in another parabola.

43. In a segment of a parabola bounded by a chord perpendicular to the axis is inscribed an ellipse touching both the curve and the chord, and having one of its axes coincident with that of the parabola. Shew that the extremities of its other axis are always situated on two parabolas, of half the dimensions of the former.

44. Two parabolas have a contact of the first order, and their diameters through the point of contact form with the tangent and normal at that point, a harmonic pencil. Shew that they will have a common chord parallel to the normal.

45. Two parabolas are drawn touching two straight lines $OAA'$, $OBB'$, in $A$, $B$, $A'$, $B'$ respectively, and $OA = OB'$, $OB = OA'$.

Shew that the three points of intersection lie in the straight line bisecting the angle $AOB$, and the straight line whose intercepts with $OA$, $OB$, are harmonic means between $OA$, $OA'$, and $OB$, $OB'$.

46. Parabolas are described touching the sides and sides produced of a triangle, so that the principal axes pass through the centers of escribed circles.

If $d_1$, $d_2$, $d_3$, be the portions of the axes intercepted between the vertices and the angles; $a_1$, $a_2$, $a_3$, each one-fourth of respective latera recta; shew that

$$\frac{d_1 d_2 d_3}{a_1 a_2 a_3} = \left(\frac{r}{s}\right)^2,$$

$r$ being the radius of the inscribed circle.

Prove that the lines joining the angles to the points of contact with the sides pass through a point.

47. Two ellipses referred, one to its axes $(a, b)$, and the other to its conjugate diameters $(a', b')$, have a common polar of the points $(\alpha, \beta)$, $(\alpha', \beta')$ respectively. Shew that

$$\frac{a}{a'} = \sqrt{\frac{\alpha}{\alpha'}}, \quad \frac{b}{b'} = \sqrt{\frac{\beta}{\beta'}}.$$

48. $S$, $S'$ are the contiguous foci of two ellipses, having the same major axis, and the excentricities $e$, $e'$, complementary, so that $e^2 + e'^2 = 1$. $B$, $B'$ the extremities of the minor axes. $CP$, $CP'$ are drawn through the centre parallel to $SB$, $S'B'$, to meet the ellipses respectively in $P$, $P'$. Prove that the tangents at $P$, $P'$ are parallel to one another.

49. $PCG$ is a diameter of an ellipse, $V$ the middle point of a chord $QQ'$ parallel to this diameter, $PV$ intersects $CQ$ or $CQ'$ in $R$, shew that the locus of $R$ is a parabola.

50. $P$ is any point in an ellipse whose foci are $S$, $H$. Shew that, if ellipses be described having $S$ and $P$ for foci, and passing through $H$, the locus of one extremity of the major axes is a circle whose radius is equal to the greatest focal distance in the original ellipse.

51. $ACA'$ is the major-axis of an ellipse, $PCP'$ any diameter, $AQ$ parallel to the conjugate diameter meets $CP$ in $Q$, $Q'$ bisects $CQ$, $A'Q$, $AQ'$ intersect in $R$; shew that the locus of $R$ is an ellipse one-third the size of the original.

52. If $CP$, $CD$ be semi-conjugate diameters of an ellipse, and lines drawn from $PD$ to the extremities of the major-axis intersect in $Q$, find the locus of $Q$, and shew that it is an ellipse.

53. From a point $T$ in the conjugate hyperbola, a line is drawn parallel to the semi-conjugate diameter $CD$, to meet the hyperbola in $U$, $V$. Shew that

$$TU.TV = 2CD^2.$$

54. $OA$, $OB$ are two straight lines, and $OA = a$, $OB = b$. $AP$, $BP$ make equal angles with $OA$, $OB$ respectively; shew that the locus of $P$ is a rectangular hyperbola whose centre is in the middle point of $AB$.

55. An hyperbola having its asymptotes parallel to the axes of co-ordinates may be made to pass through the angular points of the quadrilateral formed by the four lines whose equations are

$$(1)\quad y + x + 6a = 0.$$
$$(2)\quad y + 2x + 3a = 0.$$
$$(3)\quad y + 6x + a = 0.$$
$$(4)\quad y + 3x + 2a = 0.$$

56. If $A$, $B$, $C$, $D$ are four points on a rectangular hyperbola, such that $AB$ is perpendicular to $CD$; shew that $BC$, $AD$ and $BD$, $AC$ are also perpendicular.

57. Three hyperbolas have parallel asymptotes, shew that the three straight lines joining the points of intersection of the hyperbolas, taken two and two, all meet in a point.

58. $CP$, $CQ$ are asymptotes of a rectangular hyperbola, and $PQ$ is a tangent to the hyperbola; shew that if an ellipse be described whose half axes are $CP$, $CQ$, the common chords with the hyperbola are parallel to the diagonals of the completed rectangle $PCQR$, and that the parallelogram formed by these chords is $4\sqrt{3} \times$ area of $\triangle PCQ$.

59. If two hyperbolas have their axes parallel, the quadrilateral joining their points of intersection has the sum of its opposite angles equal to two right angles.

60. Two hyperbolas have common asymptotes occupying together all the four angles. Find the locus of the middle points of straight lines which are parallel to a given line, and have the extremities one in each hyperbola.

Trace the locus, and shew that it meets only one of the hyperbolas, in four points situated in two parallel lines.

61. Shew from general reasoning or otherwise, that the equation

$$(y^2 - 4mx)(\beta^2 - 4m\alpha) - \{\beta y - 2m(\alpha + x)\}^2 = 0,$$

represents two straight lines, and draw them.

62. If the lines represented by the straight lines whose equations are

$$(a - b')(u - bv) + (b - a)w = 0,$$
$$(b - b')(u - av) + (a - b)w = 0,$$

$u$, $v$, $w$ alone being variable, intersect in the line $u - a'v = 0$; shew that the following relation holds among the constants,

$$(a' + b')(a + b) = 2(a'b' + ab).$$

63. The inscribed circle touches the sides $a$, $b$, $c$, of a triangle in the points $a$, $b$, $c$. The circles escribed opposite to the angles $A$, $B$, $C$, touch these sides or sides produced in $\alpha\beta\gamma$, $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, respectively. Shew that $A\alpha$, $B\beta_1$, $C\gamma_2$ meet in a point, as also each of the systems

$$A\alpha, B\beta_2, C\gamma_1, \quad Bb, C\gamma, A\alpha_2, \quad Cc, A\alpha_1, B\beta.$$

64. The sides of a triangle $ABC$ are bisected forming a new triangle $A_1B_1C_1$, which is treated in the same way, and

so on $n$ times; if $\alpha, \beta, \gamma$ be the perpendiculars from a point $(x, y)$ on the sides of the triangle $ABC$, shew that the equation

$$\beta \sin B + \gamma \sin C = \frac{2^{n+1} + (-1)^n}{2^n + (-1)^{n+1}} \alpha \sin A$$

represents the straight line $B_n C_n$,

(1) when $n = 2$, (2) generally.

65. Determine the loci represented by the following equations, the axes being rectangular:

(1) $x^2 + cy - c^2 = 0$.

(2) $2x^2 - 6y^2 + xy + 2x + 18y - 12 = 0$.

(3) $2x^2 + 3xy - 2y^2 - ax + 3ay - a^2 = 0$.

(4) $\left(\frac{x}{a} + 1\right)^2 + \frac{3}{2}\left(\frac{x}{a} + 1\right)\frac{y}{b} - \frac{y^2}{b^2} = 0$.

(5) $9(x^2 + y^2) + 6xy + 4a(x - y - a) = 0$.

(6) $x^2 + 3xy + y^2 - 2ax - 4ay = 0$.

(7) $xy + ax - a^2 = 0$.

(8) $xy - x + 2y = 0$.

(9) $(x + y)^2 - 2ax + a^2 = 0$.

(10) $\sqrt{2}(x - a)(y + x) + a^2 = 0$.

66. Find the equations to the asymptotes of the hyperbolas whose equations are,

(1) $x^2 - 2xy + 4x + 3 = 0$,

(2) $y^2 - xy + ax = 0$,

the axes being rectangular.

67. Find the eccentricity and asymptotes of each of the hyperbolas,

$$4xy - 3x^2 - 2ay = 0,$$

$$y^2 - 4x^2 + 2y - 4x - 9 = 0,$$

the axes being rectangular.

68. Find the axes of the hyperbolas,

(1) $xy - ax + by = 0$,

(2) $xy - ax + a^2 = 0$,

the axes being inclined at an angle $\alpha$.

69. Prove that the angle between the asymptotes of the hyperbola

$$12x^2 - 4\sqrt{3}xy + a^2 = 0$$

is $\frac{\pi}{6}$, and find the equations to its axes.

70. Find the equation of the axis of the parabola whose equation is

$$(y - x)^2 = a(x + y).$$

71. Find the latus rectum and the co-ordinates of the vertex of the parabola whose equation is

$$(3x - 4y)^2 - 50ax + 25a^2 = 0.$$

72. Find the latus rectum, and equation to the axis of the parabola whose equation is

$$(12x - 5y)^2 - 26a(y + 8x) + 16a^2 = 0.$$

73. Shew that the latus rectum of the curve

$$x^2 + 2xy + y^2 + 2dx + 2ey + 1 = 0 \text{ is}$$

$$\frac{d+e}{(2)^{\frac{3}{2}}}.$$

74. The equation to the directrix of a parabola represented by the equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \text{ is}$$

$$2(by - ax) = \frac{b}{be - cd}(af - d^2 + cf - e^2).$$

75. If $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$ be an equation of the second order, give a geometrical interpretation to the equations,

$$ax^2 + 2bxy + cy^2 = 0 \ldots\ldots (1)$$
$$2dx + 2ey + f = 0 \ldots\ldots (2)$$
$$ax + 2by + 2d = 0 \ldots\ldots (3)$$
$$cy + 2bx + 2e = 0 \ldots\ldots (4).$$

76. Prove that the conditions that the equation

$$l\alpha^2 + m\beta^2 + n\gamma^2 + l'\beta\gamma + m'\gamma\alpha + n'\alpha\beta = 0$$

may represent a circle, are

$$lb^2 - m'ab + ma^2 = mc^2 - l'bc + nb^2 = na^2 - m'ca + lc^2,$$

and if the fundamental triangle be equilateral, these reduce to

$$l + l' = m + m' = n + n'.$$

77. Find the number of conditions that must be satisfied by the coefficients of the general equation of the $n^{\text{th}}$ degree that it may represent a series of parallel lines at equal distances from one another.

78. Two ellipses have a common focus and equal eccentricities, shew that their common chord is parallel to the line bisecting the angle between their major axes.

79. The angle subtended at the focus by the chord of a conic is $120°$, shew that the locus of intersection of tangents at its extremities is

$$l = r(1 + 2e\cos\theta).$$

80. Shew that if $PSQ$, $P'SQ'$ be two perpendicular focal chords,

$$\frac{1}{SP^2} + \frac{1}{SQ^2} + \frac{1}{SP'^2} + \frac{1}{SQ'^2} \text{ is constant.}$$

81. $A$, $B$ are fixed points in $Ox$, $Oy$, and $a$, $b$ are always taken in $Ox$, $Oy$, so that

$$\frac{1}{Oa} - \frac{1}{OA} \text{ varies inversely as } \frac{1}{Ob} - \frac{1}{OB}.$$

Shew that the locus of the intersection of $Ab$ and $Ba$ is a conic section touching $Ox$, $Oy$, in $A$ and $B$.

82. $SY$ is perpendicular from the focus in the tangent at any point $P$ in a conic section, $T$ the point where the tangent meets the directrix. Shew that the perpendicular from $Y$ on $ST$ is a third proportional to $SP$ and $SY$.

83. Two conic sections have a common tangent $AB$ at the same point $B$, and another $ACC'$, prove that the straight line joining the points of intersection cuts the tangent $ACC'$ in $D$, so that $AC$, $AD$, $AC'$ are in harmonic progression.

84. If conics be described touching two straight lines $Ox$, $Oy$ in two points $A$ and $B$ such that $OA = \frac{1}{a}$ and $OB = \frac{1}{b}$, then the centers of all these conics will lie in the straight line

$$y = x.$$

Interpret the solution $ax + by - 1 = 0$.

85. Find the equations of the tangents at the points where the conic section whose equation is

$$(ax + by - 1)(ax + b'y - 1) + cxy = 0,$$

meets the axis of $y$: and the equation of the line joining their point of intersection with the point where the conic touches the axis of $x$.

86. Shew that the tangents at the points where the conic section referred to two tangents is cut by the straight line $y = \lambda^2 x$, intersect in the straight line $ax + by - 1 = 0$, where $y = -\lambda^2 x$ cuts it.

87. $ABC$ is a triangle, $PQ$ two points in $AB$, $AC$ such that $PQ$ is always parallel to a given straight line. Shew that the locus of the intersection of $BQ$, $CP$ is a conic section circumscribing the triangle, and that $BC$ is one of its diameters.

88. The equations of three straight lines being $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and of two conics

$$\alpha\beta = \mu^3\gamma^2 \quad \text{and} \quad \alpha\gamma = \lambda^3\beta^2,$$

shew that

$$\alpha + 4\lambda\mu(\lambda\beta + \mu\gamma) = 0$$

is the equation of a common tangent to the conics.

89. Prove that if a conic be described about any triangle, and points where the lines bisecting the angles of the triangle meet the conic be joined, the intersections of the sides of the triangle so formed with the sides of the original triangle lie in a straight line.

90. Apply the method of reciprocal polars to the following problems:

(1) If two fixed tangents to a conic section be cut by a third, the points of intersection subtend a constant angle at the focus.

(2) Given, one focus, the excentricity, and direction of the axis, construct a hyperbola which shall pass through a given point.

(3) From the extremities $P$, $Q$ of a focal chord of an ellipse lines are drawn, through any point in the ellipse, to meet the directrix in $p$, $q$. Shew that the angle subtended at the focus by $Pq$ is the complement of the angle subtended by $Qp$.

91. If the chords of contact corresponding to a series of points, external to a given ellipse, pass through a point in the directrix, shew that the points all lie in a straight line passing through the focus.

Deduce this property also by means of reciprocal polars.

92. The curve $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ will be a circle if

$$\frac{l}{\sin 2A} = \frac{m}{\sin 2B} = \frac{n}{\sin 2C}.$$

93. If opposite sides and diagonals of a quadrilateral inscribed in a conic section meet in $A$, $B$, $C$, and any tangent to the conic meet the sides of the triangle $ABC$ in $A'$, $B'$, $C'$; and the lines joining $A$, $B$, $C$ respectively with the intersections of $BB'$, $CC'$; $CC'$, $AA'$; $AA'$, $BB'$ meet the opposite sides in $D$, $E$, $F$; prove that $DE$, $EF$, $FD$ will also touch the conic.

94. Shew that in general two parabolas can be described passing through the points of intersection of the curves whose equations are

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

$$a'x^2 + 2b'xy + c'y^2 + 2d'x + 2e'y + f' = 0,$$

and that if the axes of these parabolas be at right angles to each other,

$$\frac{b}{b'} = \frac{a-c}{a'-c'}.$$

# DIFFERENTIAL AND INTEGRAL CALCULUS.

1. Differentiate

(1) $x^3 - 3\log\sqrt[3]{(1+x^3)}$.

(2) $\log\dfrac{x-a}{x+a}$.

(3) $\log\dfrac{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}}{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}$.

(4) $-15(3x^{-5}-4x^{-3}+8x^{-1})(1+x^2)^{\frac{1}{2}}$.

(5) $x^{\sin x}$.

(6) $e^{\frac{\tan x}{x}}$.

(7) $(\sin x)^{\log x}$.

(8) $(\cos x)^{\sin x}$.

(9) $(\tan x)^{\cot^{-1}x}$.

(10) $\tan^{-1}x + \tan^{-1}x^3$.

(11) $\dfrac{\cos 3x + \cos x}{\sin 3x - \sin x}$.

(12) $x\sqrt{(a^2-x^2)} + a^2\sin^{-1}\dfrac{x}{a}$.

(13) $e^x\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{2}}$.

(14) $\cos\{\cos(\cos x)\}$.

(15) $\sqrt{\{\sin(x+a)\sin(x-a)\}}$.

Shew that the result of the last is

$$\cos x\sec\theta, \text{ if } \sin x = \sin\alpha\operatorname{cosec}\theta.$$

2. Investigate the differential coefficients of

$$(1)\ \log(1+\sin x) \quad \text{and} \quad (2)\ \tan^{-1}\frac{3x-x^3}{1-3x^2}.$$

Shew that the second differential coefficients of these functions are

$$(1)\ -\frac{1}{1+\sin x} \quad \text{and} \quad (2)\ -\frac{6x}{(1+x^2)^2}.$$

3. Solve the equation

$$\frac{d}{dx}\left(\sin x + \frac{1}{2}\cos 2x\right) = 0.$$

4. Find the differential coefficient of $e^x$ with respect to $\sqrt{x}$.

5. Prove that

$$(1)\ \frac{dx^n \log_e x}{dx} = x^{n-1}\log_e(ex^n).$$

$$(2)\ \frac{dx^{x^2}}{dx} = x^{x^2}\log(ex^2)^x.$$

$$(3)\ \frac{d}{dx}\sin^{-1}(3x-4x^3) = \frac{3}{\sqrt{(1-x^2)}}.$$

6. If $u = \dfrac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\sqrt{\dfrac{(a-b)}{a+b}}\tan\phi\right\}$

$$-\frac{1}{\sqrt{(b^2-a^2)}}\log_\epsilon\left\{\frac{\sqrt{(b+a)}+\sqrt{(b-a)}\tan\phi}{\sqrt{(b+a)}-\sqrt{(b-a)}\tan\phi}\right\},$$

prove that $\dfrac{du}{d\phi} = 0$.

7. Given $\cos x \cos 2x \ldots\ldots \cos 2^{n-1}x = \dfrac{\sin 2^n x}{2^n \sin x}$,

shew that

$$\tan x + 2\tan 2x + \&c. \ldots\ldots + 2^{n-1}\tan 2^{n-1}x = \cot x - 2^n \cot 2^n x.$$

8. If
$$u_n = (e^x + e^{-x})^n,$$
prove that
$$\frac{d^2\omega_n}{dx^2} = n^2 u_n - 4n.(n-1)\,u_{n-2}.$$

9. Find the $r^{\text{th}}$ differential coefficients of

(1) $x^2 \log x$.

(2) $e^{\frac{x}{\sqrt{3}}} \cos x$.

(3) $x^3 e^x$.

(4) $\dfrac{1}{x^2 - 3x + 2}$.

10. The expression for the remainder after two terms of the expansion of
$$f(x+h) \text{ is } f''(x_0 + \theta h)\,.\,\frac{h^2}{2}.$$

Shew that in the remainder after two terms of the expansion of $(x_0 + h)^3$, $\theta = \frac{1}{3}$.

11. Give a geometrical interpretation of the failure of Taylor's theorem, where
$$F(x) = x - a + 2(x-a)^2 + 3(a-x)^{\frac{5}{2}}.$$

12. Expand $e^x \cos \sqrt{3}x$ to four terms. What is the term involving $x^n$?

13. Expand $\log(\cos x)$ to four terms.

14. Expand by Maclaurin's Theorem to $x^4$
$$(e^x + e^{-x})^n.$$

15. Shew that
$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{2.3} - \frac{x^4}{3.4} \text{ \&c.} \ldots ;$$

and if $u_n$ be the coefficient of $\dfrac{x^n}{\lfloor n}$,
$$u_{n+1} + nu_n - \frac{n.(n-1)(n-2)}{1.2.3} u_{n-2} + \ldots\ldots = \cos\frac{n\pi}{2}.$$

16. If $y = m + e\sin y$, expand by Lagrange's Theorem $\log\tan\left(\frac{y}{2}\right)$, in a series ascending by powers of $e$.

17. Shew by Lagrange's Theorem that the coefficient of $x^n$ in the expansion of $y^2$ in ascending powers of $x$, where $y = xe^y$ is

$$\frac{2\,.\,n^{n-3}}{\lfloor\underline{n-2}}, \text{ when } n > 2.$$

18. Expand, by Lagrange's method, or any other, $\sin y$ in ascending powers of $x$, to $x^2$ where $y = \theta + xe^y$, and shew that the coefficient of $x^n$ is

$$\frac{1}{\lfloor\underline{n}}\,(1+n^2)^{\frac{n-1}{2}}\,e^{n\theta}\cos\{\theta + (n-1)\cot^{-1}n\}.$$

19. Find the limits of the following functions:

(1) $\dfrac{e^x - 1}{\sin 2x - \sin x}$, when $x = 0$.

(2) $\dfrac{x\tan x}{1-\cos x}$, $x = 0$.

(3) $\dfrac{x\tan 3x}{\cos x - \cos 2x}$, $x = 0$.

(4) $\dfrac{1 - x\cot x}{1-\cos x}$, $x = 0$.

(5) $\dfrac{4x\tan x - \pi}{\cot 2x}$, $x = \dfrac{\pi}{4}$.

(6) $\dfrac{2\tan x - \tan 2x}{x(1-\cos 3x)}$, $x = 0$.

(7) $\dfrac{x^3(\cot x)^2 + \sin x}{x}$, $x = 0$.

(8) $\dfrac{\sin(\tan x)}{\tan(\sin x)}$, $x = 0$.

(9) $\dfrac{\log\cos(x-1)}{1 - \sin\dfrac{\pi x}{2}}$, $x = 1$.

(10) $\dfrac{\sin x - xe^x}{\cos x - e^{2x} + 2x}$, when $x = 0$.

(11) $\dfrac{x \log(1+\sin x)}{e^{\tan x} - x - 1}$, $x = 0$.

(12) $\dfrac{e^x + \log\{(1-x)e^{-1}\}}{\tan x - x}$, $x = 0$.

(13) $\dfrac{\log\left(\sin \dfrac{\pi x}{2}\right)}{(x-1)^2}$, $x = 1$.

(14) $\dfrac{\log x - \log \sin x}{\sin x \log(1+x)}$ $x = 0$.

(15) $(1-x)e^{\frac{x}{1-x}}$, $x = 1$.

(16) $x^3 \left(\log \dfrac{1}{x}\right)^4$, $x = 0$.

20. If $x = a \cos t$, and $y = b \sin t$,
express $\dfrac{d^2y}{dx^2}$ in terms of $a$, $b$, and $t$.

21. Change the independent variable from $x$ to $t$, in

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x}, \text{ where } x = e^t.$$

22. If $R$ be a function of $x$ and $y$, and

$$x = r\cos\theta, \quad y = r\sin\theta,$$

prove that $$\left(\frac{dR}{dx}\right)^2 + \left(\frac{dR}{dy}\right)^2 = \left(\frac{dR}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dR}{d\theta}\right)^2.$$

23. If $x = r\cos\theta$ and $y = r\sin\theta$,

shew that $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt}.$$

24. If $R = f(r, \theta)$, $r = a\pi + F(\varepsilon - \pi)$, and $\theta = \varepsilon + \phi(\varepsilon - \pi)$, prove that

$$\frac{dR}{d\varepsilon} + \frac{dB}{d\pi} = a\frac{dR}{dr} + \frac{dR}{d\theta}.$$

25. If $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$, $z = r\cos\theta$, shew that

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 = \left(\frac{du}{dr}\right)^2 + \frac{1}{r^2\cos^2\theta}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{r^2}\left(\frac{du}{d\theta}\right)^2.$$

26. Transform the equation

$$\frac{dz}{dx}\cdot\frac{d^2z}{dx\,dy} - \frac{dz}{dy}\cdot\frac{d^2z}{dx^2} = 0$$

into one in which $y$, $z$ are the independent variables.

27. Find the values of $x$ which make $u$ a maximum or minimum when

(1) $u = (x-1)^2 . (x+1)$.

(2) $u = (x-a)^3 (x-2a)^2$.

(3) $u = x . (a+x)(a-x)^3$.

(4) $u = x(x-a)^{\frac{2}{3}}$.

(5) $u = (a+x)^{\frac{3}{2}} (a-x)^2$.

(6) $u = a . \cos x - a . \cos 2x$.

(7) $u = \sin x \sin(a+x)$.

(8) $u = \cos 3x - 3\cos x$.

(9) $u = xe^{-x^2}$.

28. If $x = x_0$, makes $f(x)$ a minimum; illustrate by means of a curve, whose equation is

$$y^3 = a(x-a)^2,$$

the case in which $f'(x_0) = \infty$.

29. Shew that the greatest rectangle which can be inscribed in an ellipse is half that under the axes.

30. Shew that the greatest triangle which can be inscribed in an isosceles triangle, having one side parallel to the base, is one-fourth of the given triangle.

31. Shew that the isosceles triangle, whose base is a chord of a circle and vertex in the bisection of a radius, has the greatest area when the height is equal to a radius of the circle.

32. Draw the shortest line to a parabola from a given point in its axis: and explain the result if the distance of the point from the vertex is less than the semi-latus rectum.

33. Prove that of all circular sectors described with the same perimeter, the sector of greatest area is that in which the circular arc is double of the radius.

34. Of all parabolas which can be inscribed in a circle, touching the circle at the vertex, the greatest area cut off by the chord common to the parabola and circle is that for which the chord bisects the radius of the circle.

35. An ellipse is circumscribed round a triangle whose angles are 90°, 30°, and 60°, one axis of the ellipse being parallel to the hypothenuse of the triangle; determine the ellipse so that this axis may be a minimum or maximum.

36. Shew that $\frac{m^2}{4}$ is the area of the greatest triangle which can be formed with the lines $a$, $b$, $c$, subject to the condition

$$a^3+b^3+c^3=3m^3.$$

37. Three lines $x$, $y$, $z$ are drawn from the angular points of a triangle to a point within it. Shew that when $x^n+y^n+z^n$ is a minimum we have

$$\frac{\sin(z,y)}{x^{n-1}}=\frac{\sin(x,z)}{y^{n-1}}=\frac{\sin(y,x)}{z^{n-1}}.$$

where $(x, y)$ denotes the angle between $x$ and $y$.

38. Eliminate the constants from the equation

$$(1)\quad y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)+c',$$

and the functions from the equations

$$(2)\quad z=\phi(e^x \sin y).$$

$$(3)\quad z=\phi(y^{e^x})$$

$$(4)\quad z=\phi(x+y+z),$$

$$(5)\quad z=y\,\phi\left(\frac{x}{\sqrt{(x^2+y^2)}}\right),$$

(6) $z=\phi(x^2+y^2)+\psi(x^2-y^2)$.

(7) $z=x\phi(y)+y\psi(x)$.

(8) $z=\phi\{x+f(y)\}$.

39. The curve whose equation is $u=0$ has three branches passing through the point $(a, b)$. Two of the tangents at that point are at right angles to each other. Shew that the equation to the third tangent is

$$\frac{y-b}{\left(\frac{d^3u}{dx^3}\right)_{(a,b)}} = \frac{x-a}{\left(\frac{d^3u}{dy^3}\right)_{(a,b)}}.$$

40. Find the asymptotes of the curves whose equations are

(1) $xy^4-ay^4+x^2y^3-b^5=0$.

(2) $x^4y^2-x^6+x^4y+x^5+x^3=0$.

(3) $(y^2-3xy+2x^2)x-4ay^2=0$.

(4) $x(y+x)^2(y-2x)=a^4$.

In (3) find the shape at the origin, and in (4) find on which side of its asymptotes the curve lies.

41. If the equation of a curve of the $n^{\text{th}}$ order be written in the form

$$x^nF\left(\frac{y}{x}\right)+x^{n-1}\phi\left(\frac{y}{x}\right)+\ldots=0,$$

$x^nF\left(\frac{y}{x}\right)$ containing the terms of $n$ dimensions and so on.

And if $F(z)=0$ have 2 roots each equal to $\alpha$, and $F(\alpha)$ be not zero, the asymptotes are common parabolas.

42. The curve

$$y^3-y^2-8x^6-36x^5-2x^4=0$$

has a parabolic asymptote of the second degree whose axis is parallel to the axis of $y$.

43. If $r^2=a^2\cos 2\theta$ be the equation of a curve, shew that

$$u+\frac{d^2u}{d\theta^2}\propto\frac{1}{r^5}.$$

44. The radii vectores of the points of contact of perpendicular tangents to a cardioid, whose equation is

$$r = a(1 + \cos\theta),$$

are inclined at 60°.

45. Find the point of inflexion in the curve

$$a^2 y = x(a^2 - x^2).$$

46. Find the multiple point in the curve whose equation is

$$x^4 + a^2 xy - by^3 = 0,$$

and the direction of the branches. Trace the curve.

47. Find the singular points in the curve

$$(y^2 - a^2)^3 - x^4(2x + 3a)^2 = 0.$$

48. Determine the circle $x^2 + y^2 + Ax + By + C = 0$ so that it shall have a contact of the 2nd order with the curve $y^m = ax^{m-1}$ at the point $x = a$, $y = a$.

49. Determine the curvature of the curve $ay^3 = x^4$ at the origin.

Is the curve continuous at this point?

50. Shew that the radius of curvature in the cycloid

$$= 4a\cos\phi$$

when $\phi$ is the inclination of the tangent at any point of the cycloid to the tangent at the vertex, and $a$ the radius of the generating circle.

51. At the point of contact $(x, y)$ of a curve and its circle of curvature, $\dfrac{d^4x}{ds^4}$ differs from the corresponding quantity for the circle by

$$2\frac{d^2x}{ds^2}\frac{d}{ds}\left(\frac{1}{\rho}\right) + \frac{dx}{ds}\left\{\frac{d^2\left(\frac{1}{\rho}\right)}{ds^2} + \frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}.$$

52. If $p = r\sin\alpha$ be the equation to a curve, find the equation to the evolute.

53. A tangent to the evolute of a parabola at the point where it meets the parabola is also a normal to the evolute at the point where it again meets it.

54. Find the envelope of the series of curves whose equation is

$$r = ae^{\frac{a\theta}{c}}$$

when $a$ is a variable parameter.

55. Circles are described having their centres in the circumference of a given circle and radii equal to the arcs intercepted between their centres, and a fixed point in the given circle. Find the envelope of the system.

56. A chord is drawn through the vertex of a parabola and a circle is described upon it as diameter. The locus of the ultimate intersections of all such circles is a cissoid whose asymptote is the directrix of the parabola.

57. $P$, $Q$, $R$, $S$, $T$ are five points in a curve of continuous curvature whose abscissæ are in arithmetical progression, the common difference being $\delta x$; shew that as $\delta x$ diminishes without limit, $PT$, $QS$, and the tangent at $R$ ultimately intersect in the same point; and that in the parabola $y^2 = mx$ the locus of this point is a parabola with the same vertex and axis.

58. Trace the curves whose equations are

(1) $x^2(y+x) - a^2y = 0.$

(2) $ax^2 - x^2y - by^2 = 0.$

(3) $xy^2 + x^2y = a^3 \ldots\ldots$

(4) $xy^2(x-a) = b^4 \ldots\ldots$

(5) $x^3 - x^2y - a^2x + 4a^2y = 0.$

(6) $a(y^2 - ax)^2 = x^5 \ldots\ldots$

(7) $y(x-a)^2 = x(y-2a)^2.$

(8) $(ay - x^2)^2 - y^2(b^2 - x^2) = 0,$

$$\left.\begin{array}{ll}(1) & a > b \\ (2) & a < b\end{array}\right\}.$$

$$(9)\quad (x-a)\,y^2 = 3axy + x^3,$$

and find where the last curve cuts the asymptote

$$y = x + 2a.$$

$$(10)\quad x^5 + y^5 - a^3xy = 0.$$

$$(11)\quad x^5 + y^5 - axy^3 = 0.$$

$$(12)\quad r = a\,\frac{\theta^2}{\sin\frac{\theta}{2}}\cdots$$

59. Trace the curve

$$y^2(x-b) = a^2\varepsilon^{\frac{x}{a}}(x-a),$$

$$(1)\quad a > b,$$

$$(2)\quad a < b;$$

and if, in the case $a < b$, $P$, $Q$ be the points at which the ordinates have maximum and minimum values,

$$\tan POx \tan QOx = \varepsilon^{\frac{a+b}{2a}}.$$

60. Integrate

$$(1)\quad \frac{1}{x^2(x+1)(x+2)}.$$

$$(2)\quad \frac{1}{x^2(x-1)^2}.$$

$$(3)\quad \frac{x^2+2x+4}{x^3+2x^2+4x+8}.$$

$$(4)\quad \frac{1}{(1+x^2)^2(2-x)}.$$

$$(5)\quad \frac{1}{(a^3+x^3)^{\frac{1}{3}}}.$$

$$(6)\quad \frac{1}{(1-x^2)^{\frac{5}{2}}}.$$

$$(7)\quad \frac{x^7}{(1+x^2)^{\frac{5}{2}}}.$$

(8) $\dfrac{1}{x\sqrt{(x^2+2x+2)}}$.

(9) $\dfrac{1}{(1+x)\sqrt{(4x^2+4x-3)}}$.

(10) $\dfrac{\tan^2 x}{4+\tan^2 x}$.

61. Find the area of the figure included between the parabolas whose equations are

$$\left.\begin{array}{l} y^2 = 4ax \\ x^2 = 4ay \end{array}\right\}.$$

62. Find the area included between the curves, whose equations are

$$x^2+y^2=4a^2, \quad y^2=3ax, \quad x^2=3ay.$$

63. Find the area of the curves

(1) $x^2-2xy+10y^2=a^2$

(2) $\left(y-\dfrac{x^2}{a}\right)^2=a^2-x^2$.

64. If $y^4+ax^2(2y-x)=0$ be the equation to a curve, shew that the area of the node is $\dfrac{128}{105}a^2$.

Trace the curve.

65. A circle whose centre is at the pole and radius $(a)$ cuts off from the curve $r=a(\sec 2\theta+\tan 2\theta)$ an area equal $2a^2$.

66. Find the volume and surface of the solid generated by the revolution of a given parallelogram about one of its sides; and the area of the curve $r.(\theta^2-a^2)^{\frac{1}{2}}=c$, which is cut off by the asymptote between the values of $\theta$, $\pi+\alpha$, and $2\pi+\alpha$.

67. Find the value of the integral $\iint(x^2+y^2)dxdy$ for the whole curve

$$y^2-2xy+2x^2=a^2.$$

68. The sum of the products of each element of an elliptic lamina multiplied by its distance from the focus is equal to $Ma.\dfrac{2+e^2}{3}$, $M$ being the mass, $2a$ the length of the major axis, and $e$ the eccentricity of the lamina.

69. Prove that

$$\int_0^\infty \left(\frac{1}{1^4+x^2}+\frac{1}{2^4+x^2}+\dots\,ad\ inf.\right)dx=\frac{\pi^3}{12}.$$

70. If $\phi(x)=\phi(2l-x)$, $n$ a positive integer, and

$S_r=\phi(\alpha)+\phi(\alpha+\beta)+\phi(\alpha+2\beta)$ to $r$ terms, $\phi(l)$ being the $n^{\text{th}}$ term,

$$S_{2n}=2S_n,$$

if the condition be $\qquad \phi(x)=\phi(l+x),$

$$S_n=S_{2n}-S_n=S_{3n}-S_{2n}=\dots\dots$$

# DIFFERENTIAL EQUATIONS.

1. INTEGRATE $(ax^2 + 2bxy)\dfrac{dy}{dx} + 2axy + by^2 - mx^{m-1} = 0.$

2. Find an integrating factor for the expression

$$\frac{x}{\sqrt{(y^2 - x^2)}}\left(y - x\frac{dy}{dx}\right).$$

3. Integrate the following differential equations:

(1) $m\left(x - y\dfrac{dy}{dx}\right) + n\left(y - x\dfrac{dy}{dx}\right) = 0.$

(2) $\sqrt{(1 - x^2)}\,.\,\dfrac{dy}{dx} - y = \sin^{-1} x.$

(3) $\dfrac{dy}{dx} = \dfrac{a}{x - y}.$

(4) $x\dfrac{dy}{dx} + 3y = \sin x.$

(5) $\dfrac{dy}{dx} + y\tan x = x^m \cos x.$

(6) $x\dfrac{dy}{dx} - y = xe^{-\frac{y}{x}}.$

4. Find the orthogonal trajectory,

(1) of the family represented by the equation $y = a^x$ when $a$ is the variable parameter.

(2) of a series of circles all of which touch one another at the same point.

5. The orthogonal trajectory of all curves whose differential equation is

$$x(x^3-2y^3)\,dx-y(y^3-2x^3)\,dy=0,$$

is

$$x^3-axy+y^3=0.$$

6. The equation of a family of ellipses is

$$ax^2+by^2=1$$

where $a-b=c$, a constant.

Shew that the curve which cuts all the individuals at an angle whose tangent is $\frac{y}{x}$ is

$$x=Ce^{-\frac{cy^2}{2}}.$$

7. $r=ae^{\cot\alpha.\theta}$ is the equation of a family of equiangular spirals, individuals of which are determined by particular values of $a$. Shew that the trajectory cutting all at an angle $\beta$ is another equiangular spiral whose angle is $\alpha\pm\beta$.

8. Find the singular solution of the equation

$$y-px+\frac{x^2}{4c}(p^2+1)=0.$$

9. Integrate the following equations:

(1) $y^2+2\left(\frac{dy}{dx}\right)^2-y\frac{d^2y}{dx^2}=0.$

(2) $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-\frac{dy}{dx}+2=0.$

(3) $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=x^m.$

(4) $\frac{d^4y}{dx^4}+13\frac{d^2y}{dx^2}+36y=\cos 2x.$

(5) $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-4\frac{dy}{dx}-8y=e^{-mx},$

10. Integrate the equations:

(1) $\dfrac{dx}{x}+\dfrac{dy}{y}-\dfrac{adz}{xy}=0,$

(2) $\log\left(ye^{\frac{z}{x}}\right)dx+\log\left(ze^{\frac{x}{y}}\right)dy+\log\left(xe^{\frac{y}{z}}\right)dz=0,$

(3) $y^3(1+x^2)\,dz=(2x+2x^3+y^2)\,ydx-2x^2(1+x^2)\,dy,$

and find a factor which will render integrable the expression

$$x(x^2+y^2+1)\,dx+ydy.$$

11. Integrate the partial differential equations:

(1) $p-q=\dfrac{z}{x}.$

(2) $mxp+nyq=z.$

(3) $(x+z)p+(y+z)q=x+y.$

(4) $(x^2+z^2)\dfrac{p}{x}+(y^2+z^2)\dfrac{q}{y}=\dfrac{x^2+y^2}{z}.$

(5) $\dfrac{1}{x^l}p-\dfrac{1}{y^m}q=\dfrac{1}{\dfrac{x^{l+1}}{l+1}+\dfrac{y^{m+1}}{m+1}+\dfrac{z^{n+1}}{n+1}}.$

(6) $sp-rq=0.$

(7) $ar+(a+b)s+bt=0.$

(8) $rq(1+q)-s(1+p+q+2pq)+tp(p+1)=0.$

12. A vessel is moving in a straight line down a river with a velocity $V$, and a boat starts from a point in the shore just opposite the vessel with a velocity $V'$, and pulls always towards the vessel. Find the curve described by the boat; and shew that if it overtakes the vessel at all it will be at a distance

$$a\,.\,\frac{\sqrt{V^4+V'^4-V^2\,V'^2}}{V'^2-V^2}$$

from the point of starting; when $a$ is the distance of the vessel from the shore.

13. Solve the equation

$$u_{x+3} + 3u_{x+2} - 9u_{x+1} + 5u_x = 0.$$

14. The population of a town at the end of any year can be found by subtracting eleven times the population at the end of the previous year from 10 times the population at the end of the succeeding year. Nine years ago the population was 1210, and eleven years ago 1000. Shew that it increases in geometrical progression.

15. Sum the series

$$1 + 4 + 18 + 80 + 356 + \dots \text{ to } x \text{ terms.}$$

## GEOMETRY OF THREE DIMENSIONS.

1. WHAT do the equations

$$x = a \ (1), \quad y = nz + b \ (2), \quad z = mx \ (3),$$

represent separately? What do (2) and (3) represent when taken together? What do they represent when taken all together?

2. Find the equation of a straight line which passes through a given point and is perpendicular to each of two straight lines whose equations are given.

3. Find the equation of any sub-contrary section of the cone whose equation is

$$(cx - az)^2 + c^2y^2 = r^2(z - c)^2.$$

4. If $r$ be the radius of the sphere inscribed in a tetrahedron formed by three rectangular co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, and $d$ be the distance of this plane from the origin, prove that

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

5. Find the equation of the surface generated by a straight line, which always passes through two straight lines and a circle, fixed in position, the orthogonal projections of the two straight lines on the plane of the circle being perpendicular diameters.

6. Find the equation of the surface generated by a straight line which intersects the three straight lines whose equations are

$$\left.\begin{array}{l} y=0 \\ z=c \end{array}\right\} (1) \qquad \left.\begin{array}{l} x=0 \\ y=b \end{array}\right\} (2) \qquad \left.\begin{array}{r} z=0 \\ x^2+y^2=a^2 \end{array}\right\} (3).$$

7. Find the equation to the surface swept out by a straight line which always passes through two straight lines, (not in the same plane) and meets one of them at right angles.

8. If a plane be inclined at an angle $\alpha$ to the axis of a paraboloid of revolution, find the eccentricity of the section.

9. Find the equation for a conical surface whose vertex is the origin and base an ellipse whose equations are

$$x^2 \tan^2\alpha + y^2 \tan^2\beta = c^2 \text{ and } z = c.$$

Shew that the area of the projection on the plane of $xy$ of the curve of intersection of the cone and a sphere whose center is at the vertex is to the area of the greatest section of the sphere in the ratio $\cos\alpha \cos\beta : 1$.

10. Shew that the equation of a surface generated by a circle revolving about a tangent line, coincident with $Oz$, is

$$(x^2+y^2+z^2)^2 = 4a^2(x^2+y^2).$$

11. Find the locus of the vertex of the cone, enveloping an ellipsoid, so that the plane of contact constantly touches a concentric sphere.

12. $$(1) \quad \left.\begin{array}{l} Alx + Bmy + Cnz = 1 \\ Al'x + Bm'y + Cn'z = 1 \end{array}\right\}.$$

$$(2) \quad Ax^2 + By^2 + Cz^2 = 1.$$

From every point of (1) enveloping cones are drawn to (2). Shew that the plane of contact always contains the straight line

$$\frac{x-l}{l'-l} = \frac{y-m}{m'-m} = \frac{z-n}{n'-n}.$$

13. $A$ is a fixed point, $P$ a point such that the intersection of polar planes of $A$, $P$ with respect to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is perpendicular to $AP$, shew that the locus of $P$ is a cone of the second order whose vertex is $A$.

14. If $C$ be the centre of an ellipsoid, $Q$ any point, and $CQ$ meet the ellipsoid in $P$, and $R$ be any point in a straight line through $Q$ parallel to the normal at $P$, the polar planes at $Q$ and $R$ intersect in a straight line parallel to the tangent plane at $P$.

15. An ellipsoid whose semi-axes in descending order of magnitude are $OA$, $OB$, $OC$, is cut by a plane passing through $A$ and $B$ and parallel to $OC$. If $S$, $S'$ be foci of the sections $AC$, $BC$, prove that $SS'$ is the distance of the foci of the section made by the plane.

16. Two ellipsoids have their three principal axes $2a$, $2b$, $2c$ equal, and the axis $2c$ is common to both; shew that if their other axes be not coincident they intersect in two plane curves; and if $\alpha$, $\beta$ be the semi-axes of these curves,

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

17. Shew that the tangent plane at any point whose co-ordinates are $x$, $y$, $z$ in the surface whose equation is $xyz=a^3$ cuts the axes at points whose distances from the origin are $3x$, $3y$, $3z$.

18. Investigate the locus of all tangent lines at the origin and at a point $(a, 0, a)$ in the surface given in Problem 10.

19. Shew from its equation that the normal plane at a point in a curve which is at a maximum or minimum distance from a fixed point passes through that point.

20. Find the volume included between the surface

$$y^2+z^2=4ax,$$

and the plane $x-z=a$.

21. Find the volume of the portion of the cylinder whose equation is $x^2+y^2=2ax$, contained between the plane of $xy$ and the paraboloid whose equation is

$$x^2+y^2=2a\left(z+\frac{a}{2}\right).$$

22. Find the surface of a cone whose equation is

$$x^2+y^2=(z-c)^2$$

cut off by the plane of $xy$.

23. Determine the surface represented by the equation

$$x^2+4y^2+9z^2-12yz-6xz+4xy=16.$$

24. What is the shortest distance from a point on a right cone round the cone to the same point?

25. Spheres are described having their centers on the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, and their radii proportional to the fourth root of the distances of their centers from the origin; shew that the equation of the envelope is

$$x^2+y^2+z^2=(lx+my+nz+c)^2, \text{ } c \text{ being some constant.}$$

26. Find the surface which is always touched by a plane which passes through a fixed point and makes a given angle with a fixed plane.

# ELEMENTARY STATICS.

1. $AB$, $CD$ are any two equal and parallel chords in a circle, and $P$ is a point on the circumference, half-way between $A$ and $B$. Shew that if forces represented by the lines $PA$, $PB$, $PC$, $PD$ act at the point $P$, their resultant is constant.

2. If two forces act upon a particle, find the angle at which their directions are inclined, in order that their resultant may be double of the resultant if they had acted in opposite directions. Shew that neither can be more than three times the other.

3. A uniform rod rests with its extremities on two props, find at what point a weight equal to the weight of the rod must be hung, in order that the pressures on the props may be in the ratio 2 : 1.

4. A bent lever consists of two uniform heavy beams whose lengths are as 1 : 2, find the weight which must be attached to the extremity of the shorter, in order that the arms may make equal angles with the horizon.

5. A quadrantal arc rests with the convex part on a horizontal plane, shew that if $P$, $Q$ be two weights placed at the extremities it will rest in equilibrium when turned through an angle

$$\tan^{-1}\frac{P \backsim Q}{P+Q},$$

the weight of the arc being neglected.

6. A square is capable of motion in a vertical plane round an angular point $A$, and a weight half that of the square is suspended at the angular point adjacent, find the position of equilibrium.

7. A smooth wire is bent into the form of a circle, and is supported by a small ring sliding on it and attached by a string fast to a vertical wall, against which it rests, find the inclination of the string to the wall when the tension is double the weight of the circle.

8. Two smooth cylinders, of equal radii, just fit in between two parallel vertical walls, and rest on a horizontal plane, without pressing against the walls; if a third be placed on the top of them, find the resulting pressure against either wall.

9. In the system of pullies in which each string is attached to the weight, if there be three strings, find the weights of the two lower pullies, from the consideration that, if they were interchanged, equilibrium would subsist when the power is diminished by one-half.

10. If $SY$, perpendicular on the tangent $PY$ to a parabola, be a uniform rod moveable round $S$, supported by a string $PY$, shew that the tension

$$\propto (SP)^{-\frac{1}{2}}.$$

11. An equilateral triangle is suspended from a fixed point by one of its angles, and a side of the triangle leans with its middle point against a peg vertically beneath the point of suspension; shew that the direction of the pressure on the point bisects the angle from which the triangle is suspended.

12. $OA$, $OB$ are respectively the vertical and horizontal radii of a quadrant $AB$, $P$ and $Q$ are two weights connected by a string, and $P$ hangs freely along $AO$, $Q$ rests on the convex side of the quadrant. Prove that there will be equilibrium if $\alpha = \sin^{-1}\frac{P}{Q}$, the curve being supposed smooth, and angle $AOQ = \alpha$.

13. If in problem 12 the curve be supposed rough, $\mu$ the coefficient of friction, prove that the difference between the greatest and least values of $P$ consistent with equilibrium is

$$2Q\mu\cos\alpha.$$

14. If the ratio of the greatest to the least force which acting parallel to a rough inclined plane can support a weight on it be equal to that of the weight to the pressure on the plane, the coefficient of friction will be

$$\tan\alpha\tan^2\frac{\alpha}{2}$$

when $\alpha$ is the inclination of the plane to the horizon.

15. Two inclined planes are placed on a smooth horizontal plane, and a sphere rests upon them, the inclined planes being prevented from separating by a string connecting them at the lowest points; shew that the ratio of the weight of the sphere to the tension of the string is the sum of the cotangents of the inclinations of the planes.

16. A smooth sphere rests against a vertical plane, and upon a wedge which rests on the horizontal plane supposed rough, find the angle of the wedge when there is equilibrium.

17. If three uniform rods, of equal lengths, be rigidly united, so as to form half of a regular hexagon, prove that, if suspended from one of the angles, one of the rods will be horizontal; and find the position which the horizontal rod assumes, when the joint of the rod farthest from it becomes loose.

18. In the second system of pulleys a platform is suspended from the lower block. Shew that a man of weight $W$, standing on the platform may support himself by exerting on the string a force equal to $\frac{m+1}{n+1}W$, where $n$ is the number of strings, and $mW$ the weight of the platform and lower block together.

19. Two equal smooth spheres are placed in contact, and one touches a horizontal, and the other a vertical plane, find the horizontal force which will prevent the lower from being forced out, for any position of the spheres.

If the horizontal plane be rough, find the position bordering on motion, when two-thirds of the horizontal force, found for that position in the former case, acts so as to prevent rotation.

20. A particle is placed in the middle point of a horizontal, equilateral, and triangular board, and is kept in equilibrium by three equal weights, which act on it by means of strings passing through the angular points. When the particle is moved in the direction of one of the angular points, find the force tending to restore it to its position.

If the force be half of the weights, the inclination of the strings will be $\cos^{-1}(-\frac{3}{4})$.

21. Two equal particles are placed on a parabolic arc having its axis vertical and vertex uppermost, and connected by a string which passes over a smooth peg at the focus. Shew that the particles will rest in any position.

22. A circle is capable of free motion in a vertical plane round a point in its circumference, and a weight equal to its own weight hangs by a string fixed at the point and passing over the circumference, shew that the diameter of the circle through the point is inclined to the horizon at 60°.

23. Find the centre of gravity of an equilateral triangle, from which an inscribed square has been removed.

24. A triangular lamina $ABC$ having a right angle $C$ is suspended from the angle $A$, and the side $AC$ makes an angle $\alpha$ with the horizon, it is then suspended from $B$, and the side $BC$ makes an angle $\beta$ with the horizon, shew that

$$BC^2 \tan\alpha = AC^2 \tan\beta.$$

25. A polygon one of whose sides is $AB$, is suspended first from the angle $A$, and then from the angle $B$, and it is found that the angles which $AB$ makes with the horizon in these two positions are $\alpha$ and $\beta$ respectively. Shew that the distance of the centre of gravity of the polygon from $AB$ is

$$\frac{AB}{\tan\alpha \sim \tan\beta}.$$

26. A circle has cut out from it a right-angled inscribed triangle, of which the distance of the right angle upon the hypothenuse is equal to the arc of 45°; shew that the centre of gravity of the remainder is at a distance from the centre equal to one-ninth of the radius.

27. $ABC$ is a triangle inscribed in a circle whose centre is $O$, and $F$, $G$, $H$ are the centres of gravity of the sectors $AOB$, $BOC$, $COA$; shew that

$$\frac{AB}{OF}+\frac{BC}{OG}+\frac{CA}{OH}=3\pi.$$

28. If the sector of a circle balance about the chord of the arc, prove that, $2\alpha$ being the angle of the sector,

$$2\tan\alpha=3\alpha.$$

29. From the property that a system of particles can only have one centre of gravity, shew that the straight lines joining the middle points of opposite edges of a tetrahedron, intersect, and bisect each other.

30. Find the locus of the centres of gravity of all triangles inscribed in a circle, the vertex being fixed and the base of a given length.

31. Two weights $P$ and $W$, connected by a string, rest in equilibrium on a smooth fixed hemisphere, and the radii at the points where they rest make angles of 30° and 60° with the horizon; if the positions of the weights be now changed, so that $P$ and $W$ respectively occupy the positions which $W$ and $P$ did before, find the weight which must be added to $P$ in order that the equilibrium may be preserved.

32. Two inclined planes, each of inclination 45° to the horizon, are separated by an interval, and two equal weights resting on the two planes and connected by a string are kept in equilibrium by a ring, of weight equal to either of them, which slides on the string, and hangs in the interval; prove that the string has all its portions inclined at 45° to the horizon.

If the planes be rough ($\mu$ = coefficient of friction), shew that there will be equilibrium if the portions of the string between the planes be inclined at any angle between the two

$$\operatorname{cosec}^{-1}\{(1\pm\mu)\sqrt{2}\}$$

to the horizon.

33. A heavy string hangs over the edge of a rough horizontal table; find how much can hang over without slipping.

34. A rod is capable of moving round a hinge at its middle point fixed at the vertex of an inclined plane. A weight $P$ hangs freely from one extremity, and to the other a string whose length is equal to half that of the rod is fastened, having a weight $W$ attached at the other end and resting on the plane. Shew that if $\alpha$ be the inclination of the plane to the horizon, and the rod be horizontal,

$$P = 2W . \sin^2 \alpha.$$

If the plane was rough, and the coefficient of friction such that $W$ would just rest on it without any support, then the equilibrium would not be disturbed if $P$ was increased by a weight, which bears to $P$ the ratio of

$$\cos 2\alpha : 1.$$

35. A homogeneous hemisphere is placed with its curve surface upon a rough inclined plane; assuming that the centre of gravity of a hemisphere divides the axis in the ratio of $3 : 5$, shew that $\frac{3}{8}$ is the sine of the greatest inclination of the plane consistent with equilibrium when $\mu > \frac{3}{\sqrt{55}}$.

36. Two equal rods length $(2a)$ are fastened together, so as to form two sides of a square. One of them rests on the top of a rough peg. Shew that the distance of the point of contact from the middle point of the rod is $\frac{a}{2}(1 - \mu)$, $\mu$ being the coefficient of friction.

37. A rough sphere, resting on a spherical surface, is drawn from its position by means of two unequal weights $P$, $W$, joined together by means of a string, which passes over the summit. Shew that the angle which the common normal makes with the horizon is $2 \tan^{-1} \sqrt{\frac{W}{P}}$, and the coefficient of friction $\frac{1}{2}\left(\sqrt{\frac{W}{P}} - \frac{P}{W}\right)$.

Is the equilibrium stable or unstable?

38. To a weight $Q$ placed on a rough horizontal plane (coefficient of friction $\cot\alpha$) is attached a string which passes over a pulley $C$, then over $Q$, and hanging vertically supports a weight $P$. If $\theta$ be the angle which $CQ$ makes with the horizontal plane, shew that in the limiting position of equilibrium

$$\sin(\alpha+\theta)=\frac{P+Q}{2P}\cos\alpha.$$

If we have $\cos\alpha > \dfrac{2P}{P+Q}$, $\theta$ is impossible. Explain this.

39. Two hemispheres are fixed, base downwards, upon a horizontal plane, and a rod rests at an inclination $\alpha$ to the horizon on the highest points of both, the coefficient of friction with the greater and less being $\tan\lambda$ and $\tan\lambda'$, shew that the pressures are as

$$\sin(\alpha-\lambda)\cos\lambda' : \sin(\lambda'-\alpha)\cos\lambda,$$

if the rod be on the point of sliding. Explain the result if $\alpha$ be not the intermediate to $\lambda$ and $\lambda'$.

40. Two inclined planes have a coefficient of friction $=\tan\lambda$ with the substance of which a hemispherical cup is made; supposing the planes at right angles, and one inclined at an angle $\alpha$ to the horizon, and the center of gravity of the cup in the bisection of the axis, shew that the limiting positions of equilibrium of the cup are given by the equation

$$\sin\theta = 2\sqrt{2}\sin\lambda\sin\left(\alpha+\frac{\pi}{4}\pm\lambda\right),$$

where $\theta$ is the inclination of the axis of the cup to the vertical.

41. $AB$ is a smooth vertical rod, $2n$ equal and similar rods are moveable about their extremities, which are fixed at $A$, like the ribs of an umbrella. To each of them is attached, at a point whose distance from $A$ is $(c)$, a rod without weight, the other extremity of which is supported by a small ring moveable on $AB$. Prove that the force necessary to raise the ring is independent of its position.

42. Two small rings rest on the circumference of a smooth vertical circle, being attached to the highest point by means of two strings each equal to the radius of the circle. Each of

the strings passes through another ring whose weight is double either of the others and which hangs freely. Shew that when there is equilibrium the three rings will rest in the same horizontal line.

43. A string fixed to any point $A$ passes under a moveable pully $B$, over an equal fixed pully $C$, round $B$ and $C$ again, and hanging down supports a weight. Find the position of equilibrium, and the limits to the possibility of the problem.

44. A parabolic lamina cut off by the axis is placed on a smooth horizontal plane, and a rod, capable of motion round a hinge in the horizontal plane, rests against the lamina so as to be a tangent to it.

Find the horizontal force necessary to prevent sliding.

45. An isosceles rectangular wedge has its upper portion removed by means of a plane parallel to its base, and it is then held in a position such that its base makes an angle $\theta$ with the vertical. If four equal weights be attached to a string, and the string be laid on the wedge so that one of the weights hangs freely, and each of the others rests on one of the faces of the wedge, then

$$\theta = 2 \sec^{-1} 2^{\frac{1}{4}}.$$

46. One end of a uniform straight rod rests against a smooth vertical wall. A smooth ring without weight, attached to a point in the wall by an inextensible string, slides on the rod. If $\theta$ be the angle which the rod makes with the wall when in equilibrium, and if the length of the string be $\left(\frac{1}{2m}\right)^{\text{th}}$ of that of the rod,

$$\cot^3\theta + \cot\theta - m = 0.$$

47. Two weights support each other on two smooth inclined planes which have a common vertex, by means of a string which passes over a smooth pulley at a given height vertically above the vertex; write down equations for determining the position of equilibrium, and the tension of the string.

48. A rod $AB$ placed in a smooth circle, whose plane is vertical, will be kept in equilibrium in any position by a weight applied at $A$ proportional to $\frac{BC}{AC}$, $AC$ being a horizontal chord.

49. Two equal spheres rest in a paraboloid, having its axis vertical, and touch one another in the focus; shew that the mutual pressure of the spheres is half the pressure of either sphere on the paraboloid.

50. Two rods similar in every respect are capable of motion in a vertical plane round a common fixed pivot at one extremity of each, and they are kept in equilibrium in a position inclined to the horizon by a string placed over the other ends, and kept stretched by two equal weights at its extremities. Find the position of equilibrium, and whether it is stable.

51. A series of rings $a_1, a_2, \ldots\ldots a_n$ are fixed at equal distances $(a)$ in the same horizontal straight line. $AB$ is a rod, of weight $W$, and length $na$, upon which a number of rings $b_1, b_2, \ldots\ldots b_{n-1}$ without weight, slide freely. A string fastened to the end $A$ of the rod, passes through the ring $a_1$, then through $b_1$, then through $a_2$, and so on, till after passing through $a_n$, it hangs freely through a fixed ring at the end $B$ of the rod, and has a weight $P$ attached to it. Shew that if $AB$ rests in a horizontal position, its center of gravity must divide it in the ratio of $W-P : W+P$, and that the value of $W$ must lie between $P$ and $(2n-1)P$.

52. A uniform beam $AB$, of weight $W$, and length $4a$, is moveable in a vertical plane round a fulcrum $C$ which divides it in the ratio of $3:1$. A weight $nW$ slides on a string of length $4l$, whose two ends are fastened to the points $A$ and $B$ respectively. Shew that for $n > 1$ there will be four positions of equilibrium, or only two, according as

$$l \text{ is } < \text{ or } > \frac{2na}{n+1}.$$

53. $AB$, $BC$ are two uniform rods having a smooth compass joint at $B$. $AB$ moves round a smooth hinge at $A$, and the extremity $C$ of $BC$ rests on a rough horizontal plane passing through $A$. Prove that in the limiting position of equilibrium

$$\cot BCA = 2\tan\lambda$$

where $\tan\lambda$ is the coefficient of friction.

54. Two rods have their ends joined by a compass-joint and rest in a circle whose plane is vertical, prove that if the rods are at right angles they are equally inclined to the horizon.

55. $AC$ is a smooth rod inclined at an angle $\alpha$ to the horizon, $AB$, $BC$ are equal rods joined by a compass-joint, $A$ is fixed, and at $C$ is a ring sliding on $AC$, $P$ is the weight of each rod. Shew that in the position of equilibrium, $\theta$ being the inclination of each to the fixed rod,

$$\tan\theta = \tfrac{1}{2}\cot\alpha,$$

and the pressure on the fixed rod is $\frac{3}{4}P\cos\alpha$.

56. A weight $W$ is supported by means of a string attached to the junction of two equal strings terminated by rings each of weight $P$ sliding on a smooth vertical circle, $c$ = radius of the circle, $l$ the length of each of the two strings. Shew that there are three positions of equilibrium, if $\frac{c-l}{l}$ is less than $\frac{2P}{W}$. Distinguish the stable and unstable positions.

57. Two equal uniform rods are connected by a compass-joint, the extremity of one is fixed, and to that of the other is attached a string, which is fixed to a point in the horizontal plane through the fixed extremity of the first; shew that if the rods be at right angles the mutual pressure at the joint is half the weight of either rod.

58. Two similar and equal smooth uniform rods, $AB$, $BC$, have a compass-joint at $B$, a ring without weight slides on $BC$, being attached to $A$ by a string, so that the rods can rest with their ends on a smooth horizontal plane. Shew that the mutual pressure at $B$ is perpendicular to $BC$.

59. Two rough spheres, whose centers are $A$ and $B$, are placed in contact on a rough inclined plane. If $\alpha$ be the angle which $AB$ makes with the inclined plane, shew that the equilibrium is impossible unless we have

$$\sin\alpha > \frac{W \backsim W'}{W + W'},$$

where $W$ and $W'$ are the weights of the spheres.

60. Two rods $BD$, $BC$ are jointed by a compass-joint at $B$, and are similar in every respect; $DAC$ is a string attached to $D$ and $C$, and equal in length to twice the length of either rod. A sphere is placed between the two rods, find its weight in order that the rods and string may form a square.

61. A sphere $a'Pa$ rests on an inclined plane $BP$, coefficient of friction $\mu$, supported by a smooth rod $a'aAB$, without weight, which passes freely through the center $A$, and through a hole $B$ in the plane. If $ABP = \theta$, and $\alpha =$ inclination of the plane, the limiting equation of equilibrium is

$$\tan\theta + \tan(\alpha - \theta) = \mu.$$

# ANALYTICAL STATICS.

1. IF $ABC$ be a right-angled triangle, and $ABDE$, $ACFG$ be the squares on the sides, constructed as in Euc. I. 47, prove that the resultant of forces represented by $CD$, $BF$ is parallel to a diagonal of the rectangle whose sides are $AE$, $AG$.

2. A tube in the form of a parabola is placed in a vertical plane with its axis horizontal, shew that a heavy particle within it rests at a point whose abscissa is

$$\frac{a}{\mu^2},$$

$4a$ being the latus rectum, and $\mu$ the coefficient of friction.

3. $S$ is the residence of a central force $= \mu \times$ distance, $AB$ an inclined plane, smooth and of inclination $\alpha$. Prove that if $SC$ be perpendicular to $AB$, a particle will rest in equilibrium on the plane at a distance from $C$ equal to

$$\frac{g \sin \alpha}{\mu}.$$

4. A central attractive force $= \mu \times$ distance resides in the circumference of a vertical circle, at an angular distance $\alpha$ from the highest point, shew that a particle will rest at the extremity of a horizontal diameter if

$$\mu a \cos \alpha = g,$$

$a$ being the radius of the circle.

5. The axis of a parabola is vertical, an attractive force $\frac{g}{3a} \times$ distance, resides in the vertex which is the highest point, shew that a particle will rest either at the extremity of the latus rectum or the vertex.

6. A particle is constrained to remain in the curve wh equation is $x^n + y^n = c^n$, being acted on by two forces wh vary respectively as the distances from the axes. Find th positions of equilibrium, and explain the results when $n = 2$

7. If a rigid body be acted on by four forces represen in magnitude and direction by the sides of a quadrilat figure, the axis of the resultant couple is proportional to th area.

8. Two opposite forces act upon an ellipse, in the direction of tangents at the extremity of a diameter, each force being proportional to the diameter conjugate; shew that they form a couple whose moment is constant.

9. Reduce to a single couple two couples, the forces of one of which are represented by two diagonals of opposite faces of a cube, and those of the other by the edges which do not meet those diagonals.

10. $OA$, $OB$ bounding radii of a quadrant are taken for the axes of $x$ and $y$, $BT = OA$ is a tangent at $B$, shew that the equation of the resultant of forces represented by $BT$, $BA$, and $AO$, is

$$x' + y' = 2a,$$

where $a$ is the radius.

11. Three equal rods are jointed by smooth compass-joints at the extremities so as to form an equilateral triangle. Find the direction of the pressures on the lower joints when the triangle is suspended by one angle.

12. Four equal rods are jointed together loosely so as to form a quadrilateral which is suspended by an angular point, and prevented from collapsing by means of a rod without weight, of length equal to the diagonal. Compare the tensions of the rod when acting in the place of the horizontal and vertical diagonal.

13. A uniform square $ABCD$ has one side $AB$ fixed in a horizontal position, and is kept with its plane horizontal by a support at $C$, one of the angles, find the pressures on the axis.

1. A rope is fastened to the top corner of a door, passes zontally over a pulley in the doorway, and is attached to en weight. A force acting perpendicular to the door at ottom corner keeps it open at a given angle, find the sures on the hinges.

15. Find the centers of gravity

(1) Of a sector of a circle whose density varies as the $n^{\text{th}}$ power of the distance from the center.

(2) Of a circular ring in which the density varies as the distance measured in one direction along the arc from a given point.

(3) Of the arc of a cardioid $\left(r = 2a\cos^2\frac{\theta}{2}\right)$.

16. The density of a circular disk varies as the distance from the tangent line at a point in it; shew that the distance of its centre of gravity from the tangent line is $\frac{5}{4}^{\text{ths}}$ of the radius of the circle.

17. Find the centre of gravity of the area included between the curves whose equations are

$$\left.\begin{array}{l} ay^2 = x^3 \\ bx^2 = y^3 \end{array}\right\}.$$

18. Find the centre of gravity of the area included by a loop of the curve $r = a\sin^2\theta$.

19. Find the centre of gravity of a solid generated by an isosceles triangle, which always has the same vertical angle and moves parallel to itself with the extremities of its base in the circumference of a circle to which its plane is perpendicular.

20. Find the volume generated by the revolution

(1) Of a rhombus about a line through a given angle and parallel to a given diagonal.

(2) Of the portion of a parabola cut off by a line perpendicular to the axis and equal to twice the latus rectum, round the directrix.

21. Find the volume generated by a square revolving round a line parallel to its plane, whose orthogonal projection on the square is a diagonal.

22. Find the force necessary to keep two rings on a smooth horizontal rod asunder, if connected by a string whose length is $\frac{1}{96}$th part longer than the distance between the rings.

23. The middle link of a chain slides on a smooth horizontal rod at two points of which the extreme links are fixed. Shew that if the two parts of the chain are at right angles the stress on the middle link is a mean proportional between the weight of the chain and the tension at one of the lowest points.

24. Shew that the center of gravity of any arc of the catenary lies in the vertical through the point of intersection of the tangents at the extremity.

25. The shortest string which can form a catenary by resting on two smooth pegs in the same horizontal line is $ae$, where $a$ is the distance between the pegs and $e$ the base of the Napierian system of logarithms.

26. The greatest length of a certain chain which can be suspended from one end without breaking is $l$. If a catenary be formed with a length $\frac{2l}{m}$ of this chain, shew that the greatest distance possible between the points of support (supposed in the same horizontal line) will be

$$\frac{2l}{m} \cdot (m^2-1)^{\frac{1}{2}} \log \left(\frac{m+1}{m-1}\right)^{\frac{1}{2}}.$$

Shew that $(m-1)^{\frac{1}{2}} \log \left(\frac{m+1}{m-1}\right)^{\frac{1}{2}}$ is always less than unity.

27. Shew that if a chain hangs in the form of a parabola having its axis vertical, the density varies inversely as the perpendicular from the focus on the tangent.

28. A string fixed at two points is acted on by gravity and a constant normal pressure $p$ referred to a unit of length; $\sigma$ is the weight of a unit of length, and $c\sigma$ is the tension at the lowest point. Shew that the direction of the string is vertical at a height

$$c\sigma p^{-1}$$

above the lowest point.

29. A semi-cycloid is placed with its axis vertical and vertex uppermost. A heavy chain having one extremity fixed at the vertex and equal in length to the semi-cycloid lies on it; find the tension at the vertex.

30. Two equal weights ($W$) are connected by an inextensible string and laid over a portion $AB$ of a smooth curve, the length of $AB$ being less than that of the string, and the tangents at $A$ and $B$ making angles $\alpha$ and $\beta$ with the vertical. Shew that the vertical pressure on the continuous portion of the curve is equal to

$$W(\cos\alpha \backsim \cos\beta).$$

How is the whole weight $2W$ supported?

31. Find how many coils must be taken by a string round a rough cylinder in order that a weight may support another $n$ times as great.

32. On $DC$ a smooth inclined plane a weight $W$ is placed which is supported by means of a string passing over a small rough peg at $C$, under another at $B$, and over a smooth pulley at $A$ in $DC$ produced, a weight $P$ being suspended at the other extremity. Find the greatest and least value of $P$ when $BC$ is horizontal and equal to $CA$.

33. A ring slides on a smooth parabolic arc placed with its axis vertical and vertex uppermost, and is attached to the focus by an elastic string whose natural length is equal to the semi-latus rectum. If the ring could just double the length of the string, when hanging from a point, find the angle the string makes with the axis when there is equilibrium.

34. An elastic string is fixed at one extremity $A$, and has a small ring attached to the other extremity $C$ which slides on a smooth rod $BD$ moveable round a fulcrum $B$ below $A$ and

at a distance equal to $BD$ from it. If the natural length of the string be equal to half $BD$, and the elasticity such that the weight of the rod would stretch the string half as long again, then $AC$, $AB$ and $4BC$ are in harmonic progression.

35. An endless elastic string will just reach round two pegs in a horizontal plane; a ring whose weight would double the length of the string hanging from a point is slung on it; shew that if $\theta$ be the inclination of two portions of the string to the horizon,

$$\sin 2\theta = 2(\sqrt{2}-1).$$

36. A slightly elastic string when unstretched is just long enough to reach between two hooks in the same horizontal line. Shew that a ring of weight $W$ placed at its middle point will sink through

$$a\sqrt[3]{\frac{(3eW)}{2}},$$

$e$ being the measure of the elasticity, and $2a$ the distance between the pegs.

37. A sphere is suspended by means of an elastic band passing under it and hung over a peg. A similar band passes over the top of the sphere and supports a heavy ring. The natural radius of either band is the same as that of the sphere; and its elasticity, such that the weight of the sphere would stretch a piece of it to three times its natural length. Find the equations for determining the void arcs of the circles of contact.

38. One end of a heavy elastic string is attached to the highest point of a smooth vertical quadrant of a circle, shew how to determine the length to which it will stretch,

(1) when all the string lies on the quadrant;

(2) when part of it hangs below the quadrant.

39. A length $l$ of an elastic heavy string can just stretch a small portion double its natural length. Shew that if a piece of this string be laid on a semi-cycloid of length, $c$ having its axis vertical, and the string be fastened at the vertex, the length which will just reach over the semi-cycloid is

$$c\left(1-\frac{c}{3l}-\frac{7}{60}\frac{c^2}{l^2}\right) \text{ nearly.}$$

40. Two rods are capable of turning round two hinges in a smooth horizontal plane, supposing them to rest in equilibrium on a smooth hemisphere, resting on the plane, find by virtual velocities, or by any other method, the position in which either rod rests.

41. Two particles are joined by a string, and the system is in equilibrium on the convex surface of a cycloid whose axis is vertical, convexity upwards, shew that the distances along the cycloid from the highest point are inversely proportional to the weights.

42. An isosceles triangle is in stable equilibrium on a vertical circle, if the height be less than three times the radius of the sphere, the triangle resting on its base.

43. Find the least radius of the sphere on which a hemisphere will rest in stable equilibrium.

44. If a hemispherical shell rests on a sphere of the same radius, shew that the equilibrium is what is called neuter, but that it is unstable.

45. A cone rests with the centre of its base on the vertex of a perfectly rough paraboloid of revolution whose axis is vertical. Shew that the equilibrium will be stable when the height of the cone does not exceed twice the latus rectum of the paraboloid.

46. The distance of the center of gravity of a cycloidal area whose generating circle has radius ($a$) is $\frac{7a}{6}$ from its vertex, shew that the area will rest in a position of stable equilibrium on the top of a hemisphere, if the radius of the hemisphere be greater than $\frac{28a}{17}$.

47. From a right circular cylinder of uniform density is cut an oblique cylinder by two parallel sections, making an angle $\alpha$ with its axis, the length of the cylinder so formed being equal to the diameter of the base of the original one.

If this oblique cylinder be placed with one of its ends on a rough horizontal plane, determine how a given couple must be applied so as to produce the greatest effect towards upsetting it, and shew that if $\theta$ be the angle between the vertical plane in which the couple acts, and the vertical plane containing the axis of the cylinder we have, when $\alpha$ is $< 45°$,

$$\tan\theta = (\operatorname{cosec}^2\alpha - \sec^2\alpha)^{\frac{1}{2}} \cot\alpha.$$

What will be the value of $\theta$ when $\alpha$ is greater than $45°$?

# ELEMENTARY DYNAMICS.

1. $AB = BC = CD = \&c.$, $A$, $B$, $C$, $D$ being points in a vertical straight line; if a body falls from $A$, shew that the times in $AB$, $BC$,... are as

$$1 : \sqrt{2} - 1 : \sqrt{3} - \sqrt{2} : \ldots.$$

2. A body which has fallen from rest is observed at one portion of its path to fall through $m$ feet in $n$ seconds; find the number of feet described in the next $n$ seconds.

3. A locomotive is travelling at the rate of 60 miles an hour, shew that if it were suddenly stopped, the violence of the concussion would be nearly as great as if it fell from a height of 40 yards.

4. 32.2 being the measure of gravity when a foot is the unit of space, and a second that of time,

(1) find the measure of gravity, when the unit of time is 10 seconds, and the unit of space is 2 feet.

(2) find the unit of time when the measure of gravity is 24, and the unit of space is a yard.

(3) find the unit of space, when the measure of gravity is 14, and the unit of time 5 seconds.

5. What is the inclination of an inclined plane such that the accelerating force on a body sliding down it may be 16 feet, the unit of time being a second?

6. Since $\sqrt{(2fs)}$ is the velocity generated in moving through space $s$, when there is no initial velocity, by the second law of motion, $u + \sqrt{(2fs)}$ is the velocity, when there is a velocity $u$ of projection. Point out the fallacy in this argument.

7. A number of particles are projected in all directions from a given point, with the same velocity, under the action of gravity, prove that they all lie in one sphere at every instant.

8. An engine starts a train with a pressure which continues uniform for 5 minutes, when it is found that the train is moving at the rate of 23 miles per hour. If the pressure remained uniform for 15 minutes find the velocity of the train; and, assuming that $g = 32.2$, shew that this pressure : the weight of the train as 11 : 3150.

9. $A$ is the highest point of a vertical circle, and $AB$ any chord. A circle is described on $AB$ as diameter, and a tangent is drawn at $B$ to the former circle, and meeting the latter in $C$.

Shew that the time of descent down $BC$ is constant.

10. $AB$ is the vertical diameter of a circle, through $A$ the highest point any chord $AC$ is drawn, and through $C$ a tangent meeting the tangent at $B$ in the point $T$. Shew that the time of a body's sliding down $CT$

$$\propto \frac{1}{AC}.$$

11. A body begins to slide from the highest point of a parabola having its axis vertical, shew that at any point the velocity varies as the horizontal distance described.

12. Two weights, one of which is double the other, are placed upon two smooth inclined planes, which have a common vertex, and each of which is inclined at 30° to the horizon, and are connected by means of a fine string passing over a small pulley at the highest point of the planes; find the accelerating force, and shew that if the length of the string be equal to that of either plane, and the heavier weight starts from the highest point, the two weights will reach the ground at the same time, if the string be cut when one-sixth of it has passed over the pulley.

Shew also that the times of motion before and after cutting the string are each equal to the time of a body's falling freely under the action of gravity through a height equal to the length of either plane.

13. A tangent at any point $P$ of a circle meets the tangents at the extremities of a vertical diameter in $D$ and $C$ respectively. If $t$, $t'$, $T$ be the times of sliding from rest down $CP$, $DP$, $DC$ respectively, shew that

$$t : t' :: BP : AP, \text{ and that } T \text{ varies as } tt'.$$

14. $AP$ is a circular rim against which a ball moves in direction parallel to the radius at $A$; shew that if after impact the ball moves at right angles to its former direction,

$$AP = a \tan^{-1} e^{\frac{1}{2}}$$

where $a$ is the radius and $e$ the elasticity.

15. Let $ABC$ be a triangle, $D$, $E$, $F$ the points where the circle inscribed in it meets the sides $BC$, $CA$, $AB$ respectively; shew that if a ball, of elasticity $e$, be projected from $D$ so as to strike $AC$ in $E$, and then rebound to $F$, $AE = e . CE$. If the ball return to $D$, $AB = e . AC$.

16. A ball of elasticity $e$ is projected from a point $P$ in a parabola, and impinges at the vertex on a plane perpendicular to the axis. Shew that if, after impact, the ball moves in a direction parallel to $SY$,

$$e . SP = (2 + e) AS;$$

if it passes through the extremity of the focal chord through $P$,

$$SP = (1 + e) . AS.$$

17. A body is projected from a point in a horizontal plane at an angle of $45°$, shew that the time of flight is the same as if gravity ceased to act and the body supposed perfectly elastic, were reflected at the directrix to the horizontal plane.

18. An ellipse stands in a vertical plane with its major axis horizontal. A body, acted on by gravity, is projected from one end of the major axis, so as to pass through the extremity of the minor axis, and then through the other end of the major axis. Shew that the semi-axis major is a mean proportional between the semi-axis minor and the latus rectum of the parabola described.

19. A ball is projected from a point in a horizontal plane, and makes one rebound, shew that, if the second range is equal to the greatest height which the ball attains,

$$\tan\alpha = 4e,$$

$\alpha$ being the angle of projection, and $e$ the measure of elasticity.

20. $A$ and $B$ are two balls whose relative elasticity is $\frac{5}{6}$ and the mass of $A$ is double that of $B$, if they move in opposite directions, the velocity of $B$ being double that of $A$, investigate the velocities after impact.

21. Equal rings, which are inelastic, are arranged at equal distances along a smooth horizontal rod, shew that if one of them receive any velocity, the velocities over the different intervals are in harmonic progression, and the time of reaching a distance $c$ is to the time of describing the next interval as $c$ is to double each interval, $c$ being an entire number of intervals.

22. A perfectly elastic ball is projected obliquely, and on reaching its highest point strikes directly another equal ball hanging by a string from the directrix of its path, shew that the struck ball will just reach the directrix.

23. If a ball be projected from a point in an inclined plane, in a direction such that the range on the plane is the greatest, shew that the direction of motion, on striking the plane, is perpendicular to the direction of projection.

24. If the focus of the path of a projectile be as much below the horizontal plane through the point of projection, as the highest point of the path is above it, shew that the secant of twice the angle of projection $= 3$.

25. A parabola has its axis horizontal, and a perfectly elastic particle slides down a focal distance, and rebounds at the axis, find the range, and shew that the greatest range is when the focal distance is inclined at $60^\circ$ to the axis, reckoned from the vertex.

26. $AC$ is an inclined plane, $AB$ is horizontal, angle $CAB = \alpha$ is the inclination of the plane, $AB = a$, the elasticity of a ball projected from $B$ vertically upwards towards the plane $= e = \tan^2 \alpha$; prove that, after striking the plane, the motion is horizontal, and if the velocity be that due to twice the height of the point struck, the range measured from $B = 2ae$; also the range after the rebound from the horizontal plane $= 4ae^2$.

27. A parabola is placed with its vertex uppermost, and axis vertical, shew that a perfectly elastic particle projected vertically upwards will after two rebounds at the curve descend vertically if the velocity at the first impact be that due to the latus rectum, and the point of first impact be at a distance equal to the latus rectum from the focus. Also, find the greatest height.

28. From a point in the extreme ordinate of a parabola placed with its vertex downwards and axis vertical, a perfectly elastic ball is let drop so as to impinge on the curve at one of the extremities of the latus rectum; shew that if the length of the axis be equal to half the latus rectum ($L$), the ball will return to the point from which it fell, at the end of the time

$$4\left(\frac{2L}{g}\right)^{\frac{1}{2}}.$$

29. If a ball of mass $m$, strikes another of mass $nm$ at rest, the inclination of its direction to the line joining the centres at the time of impact being $30°$, find $n$ in order that $m$ may go off perpendicular to its original direction. What is the limit of the elasticity in order that this may be the case? Explain the result when $e = \frac{1}{3}$.

30. An elastic ball strikes an equal ball at rest, the direction of motion of each ball after impact makes the same angle with that of the striking ball before impact; shew that the angle is

$$\tan^{-1} e^{\frac{1}{2}}$$

where $e$ is the elasticity.

31. Two balls move with equal velocity at right angles to each other and their directions make equal angles with the line joining the centers, they are reflected so that their directions make angles $\cot^{-1}\frac{1}{2}$, and $\cot^{-1}\frac{1}{3}$ with the same line, shew that the ratio of the masses is 8 : 9.

32. Two equal and perfectly elastic balls move along a table with equal velocities, in directions at right angles to each other, and impinge, so that the line joining their centers is immediately over the edge. If the velocity of each ball be that acquired in falling through a height equal to half that of the table, shew that the distance between the points where the balls strike the floor is double the height of the table.

33. An elastic ball is projected from the focus of an elliptic billiard-table, so as to pass through the extremity of the minor axis, and then through the extremity of the major axis. Shew that the time of reaching this point is the same as if the ball had been projected with the same velocity from the other extremity of the major axis to the opposite directrix.

34. A tube of small bore and open at both ends is bent into the form of two-thirds of the circumference of a circle and is placed in a vertical position with the line joining the open ends horizontal. If two equal balls start from the highest point, shew that after leaving the tube they will impinge at a point whose distance below the open ends equals the diameter, and if the elasticity is one-third, the distance between the directrices of the parabolas described before and after impact is one-sixth of the diameter.

35. A cannon is pointed in a direction making an angle of 30° with the horizontal plane on which it stands, and fired against a fort. It is then drawn $\frac{3}{4}$ of a mile nearer the fort, and pointed in the same direction as before, when it is observed that the ball strikes the fort in the same point as in the former case. If the greatest distance which the cannon can throw the ball is one mile, shew that the height of the point which the ball strikes is 165 feet above the horizontal plane on which the cannon stands.

36. A perfectly elastic ball is projected from the middle point of one of the sides of an equilateral three-cornered room. It strikes the two other sides and returns to the point of projection. If $a$ be the length of a side of the room, and the velocity of projection be that due to the height $\frac{5a}{4}$; shew that the ball must be projected at an angle

$$\tfrac{1}{2}\sin^{-1}\tfrac{3}{5}.$$

37. A number of particles slide down radii of a circle whose plane is vertical, starting from the circumference, the locus of a particle which has attained a given velocity is a curve whose equation is

$$(r-a)\cos\theta = b.$$

38. A perfectly elastic ball is projected from a point in a vertical wall, so as to rebound against another vertical wall, parallel to the former, find what must be the angle of projection in order that if the velocity of projection be that due to $m$ times the distance of the planes, it may return to the point of projection, or the point opposite to it in the other plane, after $m$ rebounds.

39. $P$ is the extremity of the latus rectum in a rectangular hyperbola, having its transverse axis vertical. Prove that if $t$, $t'$ be the times of sliding from rest from the centre of the circle of curvature at $P$ to $P$, and from $P$ to the centre of the hyperbola, then

$$t^2 : t'^2 :: 3 : \sqrt{2}.$$

40. $AP$, $PB$ are chords of a semicircle whose diameter $AB$ is vertical: particles falling down $AP$, $PB$ respectively start simultaneously, shew that the least distance between them is equal to the distance of $P$ from $AB$.

41. Two masses move uniformly from the point of intersection of the two straight lines which are their paths, their velocities are inversely proportional to their masses, shew that their centre of gravity describes the line bisecting the angle between them; and find with what velocity.

42. A body is projected from a given point $A$ with a given velocity and in a given direction. After a lapse of $m$ seconds another equal body is projected from the same point so that the line joining the two bodies always passes through $A$. Shew that the paths of the two bodies and that of their centre of gravity will be equal parabolas.

43. A perfectly elastic ball is projected up a plane inclined at $30^\circ$ to the horizon. If the direction of the ball's motion makes an angle of $30^\circ$ with the plane, shew that after rebounding from the plane, the ball will ascend vertically.

44. Tangents at points $P$, $Q$ in the parabolic path of a projectile meet in $T$. If $S$ be the focus, shew that the velocity due to the height $ST$ is a mean proportional, between the velocities at $P$ and $Q$.

45. A body of elasticity $e$ is projected from a point in a horizontal plane. If the distance of the point of $n^{\text{th}}$ impact be equal to four times the sum of the vertical spaces described, the tangent of the angle of projection is

$$\frac{1-e}{1 \mp e^n}.$$

46. An elastic ball is projected up a plane inclined at an angle $\alpha$ to the horizon, and after one rebound it passes through a horizontal bore at a fixed point in the plane, just large enough to admit it, shew that if $\beta$ be the angle of projection with the plane

$$\cot\beta = 2(1+e)\tan\alpha + e\cot\alpha.$$

47. A body is projected at an angle $\alpha$ to the horizon so as just to clear two walls of equal height $a$, and at a distance $2a$ from one another; shew that the range is

$$2a\cot\frac{\alpha}{2}.$$

48. $M$, $m$ are the masses of two perfectly elastic balls, and $m$ is placed at a certain distance from a vertical plane; if $M$ strike $m$ with a certain velocity in such a direction that $m$ on the rebound meets $M$ again, the motion being horizontal, shew

that, in order that $M$ may have travelled over half the distance between the original position of $m$ and the wall before meeting $m$, $M=3m$. Also find their velocities after the second impact.

49. An elliptic tube of small bore is placed with its major axis vertical, and two equal balls whose common elasticity is equal to the excentricity of the ellipse start from the extremities of the minor axis; shew that after impact each of the balls will rise to a point whose distance from the minor-axis is equal to the semi-latus rectum.

50. A room is twice as high as it is broad, shew that, if a perfectly elastic ball be projected against one of the walls with a velocity which would be acquired in falling from a height $\frac{13}{12}$ of the height of the room, it will return to the point of projection if the angle of projection be $\tan^{-1}\frac{11}{3}$, and explain how the angle $\tan^{-1} 5$ satisfies the condition.

51. From the highest point of a vertical circle, a chord $AP$ is drawn, making an angle $\theta$ with the vertical diameter $AB$. A perfectly elastic heavy body impinges on the concave circumference at $P$ in the direction $AP$, and with the velocity acquired down $AP$. Shew that if after rebounding from the curve it passes through $B$

$$\sin\theta + 2\sin 5\theta = 0.$$

Solve this equation.

52. A tube of small bore is bent into the form of a regular hexagon which is held with two of its sides vertical; if from the highest point a smooth inelastic particle be let fall, find how high it will rise after passing the lowest point.

53. Two weights $P$, $Q$ which are attached at different points of a fine string which is attached to a fixed point, are revolving uniformly with a given angular velocity round a vertical line in a state of relative equilibrium, determine the inclination of the two parts of the string to the vertical.

54. $\alpha$ is the inclination of an inclined plane to the horizon, $\beta$ that of the direction of projection of a projectile to the inclined plane, $e$ the elasticity of the plane. Shew that, if, after two rebounds, the projectile arrives at the point of projection,

$$\cot\beta = (1+e+e^2)\tan\alpha.$$

55. The *length*, *breadth*, and *height* of a room are as 8, 2, 5. From a point in one of the corners half-way between the floor and the ceiling, a perfectly elastic ball is projected against the longer of the opposite walls, and after rebounding just reaches the middle point of the ceiling. Shew that if $\theta$ be the angle of elevation at which the ball is projected, and $\phi$ the angle which the plane of motion makes with the shorter wall,

$$\cot\phi = \frac{3}{4}, \ \cot\theta = 2^{\frac{1}{2}}\cos\left(\frac{\pi}{12}\right).$$

56. $AB$ is a smooth inclined plane terminating in the smooth horizontal plane $BC$. A number of balls slide down different lengths of $AB$, and are reflected by $BC$. Shew that if they all strike $BC$ again at the same distance from $B$, their elasticities must vary inversely as the lengths of $AB$ severally described by them.

57. Two masses $m$, $m'$ are connected by an inextensible string which passes over a smooth peg, and the system is allowed to descend under the action of gravity. At the end of an interval $\tau$, a mass $\mu$ is suddenly attached to the smaller of the two masses, and this operation is repeated at the end of each succeeding interval $\tau$. Shew that if $\frac{2(m \backsim m')}{\mu}$ is an integer ($p$), the system will come to rest at end of the time $(p+1)\tau$.

# NEWTON, I. II. III.

1. FIND the limit of $\frac{1+2x}{2+x}$, (1) when $x$ is diminished indefinitely, (2) when $x$ is increased indefinitely.

2. $OA$, $OB$, and $OC$, $OD$, are two pair of perpendicular radii in a circle whose centre is $O$; if $AB$, $CD$ intersect in $P$, shew that ultimately when $C$ approaches $A$, $P$ will bisect $AB$.

3. Find the limit of

$$\frac{1.3+3.5+\ldots(2n-1)(2n+1)}{n^3},$$

when $n$ is indefinitely increased.

4. Shew that the areas of segments of a parabola whose bases contain the focus vary as the cube of the greatest breadths of the segments measured perpendicular to the base.

5. Apply Lemma II., to find the area of a curve in which the abscissa varies as the cube of the ordinate.

6. Two equal parabolas are described passing through the corners of a square, and touching at their vertices in its centre; find the centre of gravity of one of the portions included between them and a side of the square.

7. Find the volume generated by the revolution of a rectangle about one of its diagonals.

8. $BAC$ is an arc of continued curvature, $BA$ and $CA$ are in a constant ratio, and the tangent at $A$ cuts the chord $CB$ in $D$. Shew that ultimately, when $B$ and $C$ approach $A$,

$$DB : DC :: AB^2 : AC^2.$$

9. Shew that the locus of the foci of all parabolas having the same curvature as a curve at a given point, is a circle.

10. A circle touches a parabola in $P$, and its axis in the focus. Prove that the radius of curvature at $P$ is four times the radius of the circle. Also that the chord of the circle of curvature which touches this circle, at the point where the normal meets it, is to the latus rectum as 4 : 3.

11. If the circle of curvature at a point $P$ of a parabola passes through the other extremity of the focal chord through $P$, and the tangent at $P$ meets the axis in $T$, then the triangle $PST$ will be equilateral, and the distance of the center of the circle of curvature from the directrix will equal the latus rectum.

12. Half the chord of curvature through a focus of an ellipse is a harmonic mean between the focal distances.

13. If $G$ be the foot of the normal to an ellipse at $P$, and $GK$ perpendicular to $PG$ meet $PS$ in $K$, shew that $2PK$ is the chord of curvature at $P$ through $S$.

14. A body describes an ellipse round a force in the center. If $MPQ$ be an ordinate meeting the auxiliary circle in $Q$, $M'P'Q'$ another ordinate meeting the circle in $Q'$, prove that the time of describing the arc $PP'$ varies as the angle $QCQ'$.

15. A particle describes a parabola round a force in the focus. $A$ is the vertex, $L$ the extremity of the latus rectum, $P$ a point whose distance from the axis is the length of the latus rectum. Prove that the time in $AL$ : time in $LP$ :: 2 : 5.

16. A body revolves in an ellipse round a force in the center, and $PQ$ is a chord equal to half the axis major and parallel to it. Shew that the times of moving from $A$ to $P$, and from $P$ to $Q$ are equal.

17. Two bodies, under the action of the same centre of force in the focus, start from opposite extremities of the major axis and describe the same ellipse. Prove that the ratio of the times of arriving at the extremity of the latus rectum is

$$\frac{a\cos^{-1}e - be}{a\cos^{-1}e + be},$$

$e$ being the eccentricity of the ellipse.

18. A body describes a parabola round a force in the focus. If $v$, $v'$ be the velocities at the extremities of any focal chord, $V$ at the extremity of the latus rectum, then

$$v^2 + v'^2 = 2V^2.$$

19. The Moon moves round the Earth in 27 days 7 hours nearly, gravity at the Earth's surface will draw a body 16 feet from rest in one second nearly, the radius of the Earth is 4000 miles nearly; find the distance of the Moon approximately.

20. Compare the tension of a string to which a weight is attached which revolves round a point in a horizontal circle ten times in a second with the weight itself, the string being a yard long.

21. A weight is placed on a smooth inclined plane, to which it is attached by a string just capable of supporting a weight double the weight on the plane; find the greatest inclination of the plane, in order that the string may bear the weight, just revolving completely round the point of attachment.

22. Shew that in the elliptic orbit described under the action of a force tending to a focus, the angular velocity round the other focus varies inversely as the square of the diameter parallel to the direction of motion.

23. A string passes round two pegs in a horizontal plane, and two equal rings sliding on it are projected from the point when they are equally distant from each peg with equal velocities, so as to keep the string tight; shew that each describes a parabola with uniform velocity, and find the tension of the string for any position.

24. A ring slides on a string which is placed on a smooth horizontal table, round two pegs; shew that the ring, if projected so that the strings are tight, will describe an ellipse with uniform velocity, and that the tension of the string varies inversely as the product of the distances from the pegs.

25. Shew that if a body revolves in a circle round a center of force in the circumference, the time in a quadrant commencing from the extremity of the diameter through the center of force is

$$R^3(\pi+2)\sqrt{\frac{2}{\mu}},$$

where $\mu$ is the absolute force, and $R$ the radius of the circle.

26. Shew that when an ellipse is described under the action of a force tending in a direction perpendicular to the major axis, the velocity varies as the secant of the angle which the direction of motion makes with the major axis.

27. A body is revolving in a circle round a force in the center, find the new orbit described if the force be suddenly transposed to the further extremity of the diameter passing through the body. Force $\propto$ distance.

28. Two small equal elastic balls revolve in a circle in opposite directions about a force in the center which varies as the distance. Shew that if $\sin\alpha$ be the elasticity of the balls, then $\cos\alpha$ will be the excentricity of the ellipse in which they move after impact.

29. A body is describing a circle under the action of a force which tends to the centre, and which varies as the distance; shew that, to whatever point the centre of force be transposed, the difference of the semi-axes of the new orbit is the distance of that point from the centre of the circle.

30. A particle is revolving in a circle round a force which varies as the distance, the center of force is suddenly transferred to the extremity of the diameter through the particle and becomes repulsive; shew that the eccentricity of the new orbit is $\frac{\sqrt{5}}{2}$.

31. A body is revolving in an ellipse, about a force tending to the center, and when it arrives at an extremity of one of equal conjugate diameters, the force becomes repulsive; shew that the transverse axis of the hyperbolic orbit is a geometric mean between the axes of the ellipse.

32. A particle is describing an elliptic orbit round the centre $C$, and when it arrives at $B$, the extremity of the minor axis, the force is suddeuly transferred to the focus $S$, shew that the major axis of the new orbit bisects the angle $BSC$, and that if $a$, $b$, $\alpha$, $\beta$ be the semiaxes of the old and new orbits,

$$a-b : a+b :: (\alpha-\beta)^2 : (\alpha+\beta)^2.$$

33. A body is revolving in an ellipse round a force in the center, and when it arrives at the extremity of the major axis, the center of force is transferred to the further focus, shew that the (eccentricity)$^2$ of the new orbit

$$\frac{2e}{1+e},$$

$e$ being that of the old orbit.

34. If a particle describe an ellipse under the action of a force tending to the focus, and $v$, $v'$ be the velocities at two points equally distant from the axis on the same side, $V$ the velocity at the extremity of the minor axis; prove that

$$vv' = V^2.$$

35. A body is projected with the velocity equal to that in a circle at the same distance from a center of force varying inversely as the square of the distance, shew that the cosine of the angle of projection is the eccentricity of the orbit described.

36. Prove that in order that an ellipse, described under the action of a force tending to a focus, may have a velocity at some point equal to $\frac{1}{\sqrt{2}} \times$ velocity in a circle at the same distance, it is requisite that $e$ be not less than $\frac{1}{2}$.

37. A body describes an ellipse round a center of force in the focus, and when it arrives at the extremity of the minor axis, the central force is suddenly transferred to the center; shew that the eccentricity of the new orbit is

$$1 - \frac{BC}{AC},$$

or the ellipticity of the old orbit.

If the change took place at the extremity of the major axis, the eccentricity of the new orbit would be

$$\frac{SH}{AS}.$$

38. The velocity in a parabola round the focus is suddenly diminished in the ratio of $\sqrt{2} : 1$, shew that the semi-major axis is $SP$, and the semi-minor axis is a mean proportional between $SP$ and $AS$.

39. If at any point of an elliptic orbit described round the focus, the force suddenly become repulsive, remaining of the same magnitude, shew that the focal distance is as a harmonic mean between the axes of the elliptic and hyperbolic orbits.

40. Two small equal elastic balls are revolving in equal parabolas turned in opposite directions round a common focus, having the direction of their axes coincident; they impinge so that the line joining their centers is parallel to the axis; shew that the major axis of each of the orbits described after impact is to the latus of the parabolas as

$$1 : 1 - e^2$$

where $e$ is the elasticity.

41. A body describes a hyperbola, under a repulsive force tending from the farther focus, and when it arrives at the vertex, the force suddenly becomes attractive; shew that, if the new orbit be a parabola, $e'$ the eccentricity of the hyperbola $= 3$, if the new orbit be an ellipse of eccentricity $e$,

$$e' \pm e = 2.$$

42. A particle is describing an ellipse of eccentricity $e$ round a center of force in one of the foci. When the particle is at its least distance the force is suddenly diminished to $\left(\frac{1}{n}\right)^{\text{th}}$ of its former value. Shew that the subsequent orbit will be an ellipse, parabola, or hyperbola, according as $n$ is less, equal to, or greater than

$$\frac{2}{1+e}.$$

43. A particle is projected round a center of force which varies inversely as the square of the distance, with a velocity which is to the velocity in a circle at the same distance as $\sqrt{5} : 2$, and at an angle whose sine is $\frac{2}{\sqrt{5}}$, shew that the eccentricity of the orbit described is $\frac{1}{2}$, and that the major axis is perpendicular to the distance of projection.

44. The ratio of the axes of the Earth's and Venus' orbits is 18 : 13; find the periodic time of Venus.

45. A body revolves in an ellipse round a force in the focus. If $\alpha$, $\beta$ be the angular velocities at the extremities of any chord parallel to the axis major, the periodic time will be

$$\frac{\pi b}{2a} \cdot \left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}\right)^2.$$

46. When a second's pendulum is carried to the top of a mountain, 43 oscillations are lost in a day; prove that the height of the mountain is about 2 miles.

47. A particle is describing an ellipse round a force in the centre, and when it arrives at the extremity of the minor-axis, the force is replaced by one which is equal in magnitude at that point, but varies inversely as the square of the distance; shew that the eccentricity of the new orbit is

$$\frac{CS^2}{BC^2}.$$

48. $C$, $C'$ are the lengths of the chords of curvature drawn parallel to the asymptotes at any point of a rectangular hyperbola; $D$ the diameter of curvature; $D'$ the distance between the foot of the normal and the centre of curvature. Prove that

$$C^2 + C'^2 = D^2,$$
$$(C - C')^2 = 2DD'.$$

49. One end of a string is attached to the vertex of a smooth cone which stands with its axis vertical, and the other to a particle which revolves in a circle on the surface of the cone. If $2a$ be the length of the string, $2\alpha$ the vertical angle of the cone, and the velocity be that which would be acquired in dropping from rest through a height $a \operatorname{vers} \alpha$, the tension of the string will be equal to weight of the particle.

50. A particle describes an ellipse about a force tending to the center, when it arrives at any point $P$, the force is suddenly transferred to the focus $S$; shew that if $e$, $\varepsilon$ be eccentricities of the old and new orbits,

$$1 - \varepsilon^2 = \frac{1-e}{1+e},$$

and that the semi-axes bisect the angles $PSA$, $PSA'$, and are respectively mean proportionals between $SP$, $SA$, and $SP$, $SA'$.

51. If from every point of a hyperbola described under the action of a force in the farther focus a particle moved from rest under the action of the force at that point continued constant, until it acquired the velocity of the particle moving in the hyperbola; find the locus of the particles. If $r$, $r'$ be the radii vectores for the hyperbola and locus,

$$2ar' = r^2.$$

52. A body is revolving in an ellipse under the action of a force tending to the focus, and when it arrives at the nearest focal distance, the force is transferred to the point of intersection of the nearest directrix and the axis. Shew that the eccentricity of the hyperbola which will be described is the reciprocal of that of the ellipse, and find the transverse axis of the hyperbola.

53. If $PD$ is perpendicular on the directrix from any point of an elliptic orbit described by a particle about the focus $S$, and when the particle is at $P$, the force suddenly tends to $D$ instead of $S$, prove that the new orbit may be a parabola if $e > \frac{1}{3}$, and that in this case $SP$ passes through the intersection of the two circles, one described on $SH$ as diameter, and the other with center $S$ and radius $SA$, the shortest focal distance.

54. A body revolves in a parabola round a force in the focus, and when it arrives at a point $P$ whose distance from the axis equals the latus rectum, the force is suddenly transferred to the other extremity of the focal chord through $P$. Shew that the new orbit will be an hyperbola whose axes are as 2 : 1, and that the conjugate axis and the tangent at $P$ are inclined at equal angles to $SP$.

# DYNAMICS OF A PARTICLE.

1. PROVE that if in Problem 3, *Analytical Statics*, the particle be disturbed, the time of a small oscillation is $2\pi\,\mu^{-\frac{1}{2}}$.

2. If in Problem 20, *Elementary Statics*, the disturbance be very small, find the time of a small oscillation.

3. Two forces $\mu D$ and $\mu' D$ reside in fixed points, and a particle is projected in any direction; prove that the periodic time is $\dfrac{2\pi}{\sqrt{(\mu + \mu')}}$.

If there be any number of forces it is $\dfrac{2\pi}{\sqrt{\{\Sigma(\mu)\}}}$.

4. Two particles of mass, $m$, and $m'$, attract one another with a force which varies as the mass directly and the $(\text{distance})^2$ inversely, they are separated by two smooth parallel planes, and are slightly out of the position of equilibrium, prove that the time of a complete oscillation is

$$2\pi \frac{a^{\frac{3}{2}}}{\sqrt{(m+m')}}.$$

5. A particle is attracted by a force which varies as the distance from a point $S$, and also repelled by a force which varies as the square of the time, find the position at any time supposing the particle at $S$ in the beginning of the motion. *Deduce* from the result the distance described if the repelling force were the only force.

6. Two masses $m$ and $m'$ are connected by an inextensible string, and laid over a double inclined plane of mass $m + m'$, which is capable of moving freely upon a smooth horizontal

plane. If the section of the inclined plane be isosceles, and $\alpha$ the inclination of its sides to the horizon, the system may be kept in a state of relative equilibrium by the force

$$2(m \backsim m')\, g \tan\alpha$$

applied to the plane.

7. $CB$ is the horizontal base, $AB$ the vertical axis of a cycloid, $PQ$ is a line unwrapped from $PA$, shew that the time of sliding from $P$ to $Q$ is always the same, however far the line be unwrapped.

8. If a particle move from rest at $A$, under the action of an attracting force whose accelerating effect varies as the distance from a fixed point $S$, prove that the time of arriving at any point $P$ varies as the angle $ASQ$ of the circle whose center is $S$, radius $AS$, $QP$ being perpendicular to $SA$.

9. If $r^3 = a^3 \cos 3\theta$ be the polar equation to a curve, shew that the force tending to the pole by which a particle describes the curve

$$\propto \frac{1}{r^9},$$

and that the velocity

$$\propto \frac{1}{r^4}.$$

Find the law of force tending to the pole by which a particle describes the spiral

$$r = ae^{\theta \cot \alpha}.$$

10. Prove that if in an orbit described round a centre of force the velocity at every point is $n$ times that in the circle round the center of force at the same distance,

$$p^{n^2} \propto r.$$

What is the curve when $n = \sqrt{2}$?

11. A particle moves in a cycloid in such a manner that its velocity perpendicular to the axis is constant, shew that the velocity varies inversely as the distance along the arc from the vertex. Find the law of force under which it moves.

12. A particle is attracted to a fixed point by a force which at a distance $r$ is equal to $\mu \dfrac{3a^8 + 3a^4r^4 - r^8}{r^7}$; and is projected from a point at a distance $a$ from the centre with a velocity equal to that in a circle at the same distance, and in a direction making an angle $\tan^{-1}(\frac{1}{2})$ with the distance: determine the orbit described.

13. A particle attached to a string which wraps round a cylinder, whose axis is horizontal, commences to move from the lowest point of the cylinder, shew that it comes to rest when the string has turned through an angle $\theta$ given by the equation

$$\theta = \tan\frac{\theta}{2}.$$

14. A body is suspended from a fixed point by an elastic string, which is stretched to double its natural length when the body is in equilibrium.

Find how much the body must be depressed so that when let go, it may just reach the point of suspension; find also the time of reaching it.

15. A particle moves in a circular tube, under the action of a force which tends to a point in the tube and whose accelerating effect varies as the distance, shew that, if the particle begins to move from a point at a distance from the center of force equal to the radius, there is no pressure on the tube at an angular distance from the center of force equal to

$$\cos^{-1}\tfrac{2}{3}.$$

16. An ellipse is placed with its minor axis vertical. A smooth body slides down the curve from one extremity of the axis major. Shew that the points where the pressure on the curve is greatest, are those whose distances from the axis major are a third proportional to $SC$ and $BC$.

17. A particle begins to move from any point in a smooth elliptic tube to the focus of which a force tends, whose accelerating effect varies inversely as the square of the distance, prove that at any point the pressure on the curve varies inversely as the radius of curvature at that point.

18. Two particles $m$ and $m'$ are connected by an inextensible string equal to a quadrant of a vertical circular tube in which they are placed together at the highest point, if $m$ is slightly displaced, find the velocity of $m'$ when the string becomes tight, and shew that $m'$ will produce no pressure if

$$m' : m :: \sqrt{2} - 1 : 1.$$

19. A ring slides on an elliptic arc, and an elastic string whose natural length is less than the major axis, passes through the ring and is fixed at the foci, shew that the pressure on the arc varies as the sum of the distances of the foci upon the tangent at the position of the ring.

20. A ring of radius $a$ is made to revolve with an angular velocity $\omega$, on a horizontal smooth plane, and its substance has cohesive power sufficient to enable it to support a length $l$ without breaking. Shew that it will fly to pieces if the angular velocity be greater than

$$\frac{(gl)^{\frac{1}{2}}}{a}.$$

21. A weight $P$ is placed on the convex circumference of a smooth vertical circle, and is kept at rest by a string which passes (in a straight line) over a smooth peg at $A$ the highest point of the circle, and hanging vertically supports an equal weight $Q$ at its other extremity. If the weight of $Q$ be now increased by $\left(\frac{1}{n}\right)^{\text{th}}$ and the system left to the action of gravity, shew that it will come to rest after $P$ has described an arc

$$2\left(\frac{2\pi}{3} - \alpha\right),$$

where $\cos\alpha = \frac{n-1}{2n}$.

22. A right cylinder on a circular base is cut by any number of planes passing through the same point in its axis. In the common center of all these sections is placed a force varying as the distance. If each curve of intersection be described freely by a particle under the action of this force,

shew that if at any instant all the particles be situated on a generating line of the cylinder, they will be so throughout the whole motion.

23. Three particles are placed, one at each of the points of contact $P$, $Q$ of two tangents to a parabola, and the third at their intersection $T$. They move towards the focus $S$, under the action of a force varying as the distance. If $v_1, v_2, v$ be the velocities at $S$, of the particles from $P$, $Q$, $T$, then

$$v^2 = v_1 v_2.$$

Also if the force varies inversely as the distance, and $t_1, t_2, t$ be the corresponding times in $PS$, $QS$, $TS$, then

$$t^2 = t_1 t_2.$$

24. A particle is constrained to move in a circular tube radius $(a)$, and is attracted, force varying as the distance, by another particle, moving in another concentric circular tube radius $(b) < (a)$, with constant velocity $= b\sqrt{\frac{(2\mu b)}{a}}$. They start from the extremities of diameters, at right angles to one another. Shew that they will be in a common radius in time

$$\log(\sqrt{2}+1)\sqrt{\frac{a}{\mu b}}.$$

25. A string is wrapped round a fixed cylinder (radius $a$) standing on a smooth horizontal table, and a particle of weight $W$ attached to the string is projected from the surface of the cylinder, perpendicular to it, with velocity $\sqrt{(ag)}$; if the string breaks when the straight part has revolved through an angle $\alpha$, shew that it is of strength sufficient to bear a weight $\frac{W}{\alpha}$.

26. A weight $P$ lies upon a rough horizontal plane (coefficient of friction $\mu$). A string attached to $P$ passes through a hole in the plane and has a weight $Q$ attached to its other extremity. Shew that if the weights of $P$ and $Q$ be as 3 : 1, the greatest angle $(\alpha)$ through which $Q$ can oscillate (in one plane) on either side of the vertical is given by the equation

$$\cos\alpha = \tfrac{3}{2}(1-\mu).$$

27. A particle begins to fall towards a centre of force varying as $\frac{1}{(\text{distance})^2}$, the initial distance being $2a$. When it is at a distance $a$ from the centre, it is deflected by an indefinite plane placed at an angle of $45^\circ$ to the line of its motion. Supposing the elasticity of the particle to be perfect, shew that its greatest distance from its initial position is $2a\sqrt{2}$, and that the length of the path described between these points is

$$\frac{\pi+4}{2}a.$$

28. At the center of gravity $O$ of an equilateral triangle $ABC$ is situated a center of force varying as $\frac{1}{(\text{distance})^2}$. A perfectly elastic body begins to fall towards $O$ from a distance equal to the height of the triangle, and in a direction parallel to the side $BC$. Shew that after rebounding from $AC$ or $AB$ it will describe an ellipse having the angle $A$ for one of its foci.

29. Two rings $A$, $B$ of equal weights slide on a smooth horizontal rod and are connected by an inextensible string, on which slides another weight whose mass is double that of either of the others. $A$ and $B$ are held so that the two halves of the string make each an angle $\alpha$ with the vertical. If they be now left to move freely, shew that the tension of the string is instantaneously diminished in the ratio of

$$\cos^2\alpha : 1.$$

30. A smooth wire in the form of a circle is made to revolve uniformly with angular velocity $\omega$ about a point $A$ in its circumference in a horizontal plane. A thin ring $P$ slides on the circle and is initially at rest at the farthest distance $c$ from the fixed point, shew that the distance from $A$ at the time $t$ is $\frac{2c}{e^{\omega t}+e^{-\omega t}}$, and if $PT$ be a tangent to $P$'s path,

$$OPT=\tfrac{1}{2}POA.$$

31. Two equal heavy rings slide on a smooth vertical circle, and are connected by a horizontal extensible string whose unstretched length is $l$. If $2\alpha$ be the angle subtended by the rings at the centre of the circle when they are in equilibrium, shew that if both the rings be depressed through an equal small space, the length of the isochronous simple pendulum will be

$$\frac{(2a\sin\alpha - l)\,a}{l - 2a\sin^3\alpha},$$

whenever $l$ is greater than $2a\sin^3\alpha$.

32. There are three equal weights $A$, $B$, $C$. $A$ is connected with $B$ by an inextensible string, and with $C$ by an extensible string whose length when unstretched is $a$, and when stretched by a force equal to two of the weights is $4a$. $A$ and $C$ are placed on a smooth horizontal table, and $B$ hangs over the edge, the string lying entirely in a vertical plane perpendicular to the edge of the table. The system is then left to itself. Shew that the greatest elongation of the string connecting $A$ and $C$ is $2a$, and that the tension is zero at intervals of time equal to

$$2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

33. A uniform rod of length $a$ lying on a smooth horizontal table is constrained to move about an extremity with uniform angular velocity; a particle is attached, by a string of length equal to that of the rod, at the other end, being at rest when the rod and string are in one straight line.

Shew that when the rod has moved through an angle $\theta$, the distance of the particle from the extremity about which the rod turns, is given by the equation

$$r\,(e^{\theta} + e^{-\theta}) = 4a.$$

# RIGID DYNAMICS.

1. FIND the moment of inertia

(1) of a square about a diagonal.

(2) of a circular arc round its chord, the density being uniform, or varying as the distance, from the middle point estimated along the arc.

(3) of a parabolic area terminated by the latus rectum about the latus rectum.

(4) of a loop of the curve whose equation is $r = a\cos 3\theta$ about an axis through the pole perpendicular to its plane.

(5) of a lemniscate round an axis through the node perpendicular to its plane, the equation of the lemniscate being

$$r^2 = a^2 \cos 2\theta.$$

2. An axis passing through the centre of an ellipse is inclined at an angle of 30° to its plane, and makes equal angles with its axes. Shew that the radius of gyration about an axis through the centre, perpendicular to its plane, is eight times that about this axis.

3. Prove that if $A$, $B$, $C$, be the moments of inertia about 3 principal axes through any point of a plane area of which that corresponding to $A$ is perpendicular to the plane of the lamina, the moment of inertia about any axis inclined to the axes corresponding to $B$ and $C$ at angles $\beta$ and $\gamma$ is

$$B\sin^2\gamma + C\sin^2\beta.$$

4. Find the principal axes through the vertex of the parabolic area included between the axis and semi-latus rectum; shew that one is inclined to the axis at an angle

$$\tfrac{1}{2}\tan^{-1}\frac{35}{13}.$$

5. Find the condition that a circular plate may just revolve about a horizontal tangent as axis, and compare the pressures in the highest and lowest positions.

6. An equilateral triangle revolves round one side, which is horizontal and fixed, under the action of gravity; supposing the plane originally horizontal, compare the pressure on the axis, in this and the lowest positions, with the weight of the triangle.

7. Find the time of oscillation of two circular disks having a common tangent line, rigidly attached so that their planes are coincident: the axis of suspension being parallel to the common tangent line. Is the time affected if their plane be not perpendicular to the plane containing the tangent line and the axis of suspension?

8. A wire is bent into the form of an arc $\frac{\pi}{3}$ of a circumference, and its bounding radii. It is suspended from the centre, and is capable of motion about it in its own plane. If the wire receive a small displacement, find the time of an oscillation.

9. Find the centre of percussion for a parabolic area cut off by the latus rectum, supposed a fixed axis.

10. An isosceles right-angled triangle has its right angle fixed, if it be struck perpendicular to its plane at one of the angles of its base, find the line about which it will begin to revolve.

11. A rectangle is struck by an impulse perpendicular to its plane, find the axis about which it will begin to revolve, and determine its position with reference to an ellipse inscribed in the rectangle.

12. If a point in an ellipse's perimeter be fixed, and a blow be given perpendicular to the plane at the extremity of the diameter conjugate to that through the fixed point, shew that the impulse on the fixed point is one-fifth of the magnitude of the blow.

13. A square is supported by two pegs at the extremities of a diagonal, in a vertical position, find the initial pressure on the remaining peg, when one is suddenly removed.

14. A sphere is placed upon two smooth equal spheres held in contact, which rest on a smooth horizontal plane, in the position of equilibrium, shew that if the spheres be left to themselves the pressure on the upper sphere is instantaneously diminished in the ratio of 7 : 6.

15. A rod rests between a smooth horizontal and vertical plane being prevented from sliding by a string reaching from the angle to the lowest point; shew that the ratio of the pressures on the vertical plane, before and after the string is cut, is

$$3 \sin^2\alpha : 2,$$

$\alpha$ being the inclination of the rod.

16. If a billiard ball resting on a rough table be struck at any point, prove that it will move in a straight line.

17. If one billiard ball strikes another with a given angular velocity, prove that the angle at which the striking ball begins to move is less, if the angular velocity be such as to increase the velocity of the point of contact of the balls, than if it be in the contrary direction, *cæteris paribus*.

18. A sphere has an angular velocity about a horizontal diameter, and falls upon a rough inelastic board which is moving uniformly in a horizontal plane in direction of this diameter, find the initial direction of motion of the sphere and its path afterwards.

19. A sphere rests on a rough fixed sphere of the same magnitude, at the highest point, shew that if slightly disturbed it will roll down.

20. Write down the equation derivable from the Principle of Conservation of Areas in each of the following cases:

(1) two particles are attached by a string, and one slides down a vertical tube, and the other is projected in any direction.

(2) four equal rods are jointed, so as to join a parallelogram, and are rotating about one of the angular points in a horizontal plane.

21. Write down the equation of Vis Viva in the following case:

$A$ and $B$ are two rings sliding on a horizontal rod supporting a third ring which slides on a string attached to $A$ and $B$.

22. Write down the equation of Vis Viva when a rod $BC$ is moveable about an extremity, and a weight $P$ is attached to the other end by a string passing over a pulley $A$, in the same horizontal plane as $B$ and placed so that $AB = BC$. Deduce the time of a small oscillation about the position of equilibrium.

23. If the string in the last problem be elastic, and fixed at $A$, apply Vis Viva to obtain the equation of motion.

24. A circular ring is fixed in a vertical position upon a smooth horizontal plane, and a small ring is placed on the circle, and attached by a string, which subtends an angle $\alpha$, to the highest point; prove that, if the string be cut, and the circle left free, the pressures on the ring before and after the string is cut are in the ratio

$$M + m \sin^2 \alpha : M,$$

$m$ and $M$ being the masses of the ring and circle.

25. Explain why in playing at cup and ball we spin the ball in order to catch it on the point, and not when we would catch it in the cup.

26. A small ball is supported in the air, by means of a thin vertical jet of water. It is observed that the ball remains in the air, even when the direction of the jet does not pass through its centre of gravity; explain this.

27. If $BC$ be a uniform rod suspended from $A$, by means of a string $AB$, and the system be whirled round with uniform angular velocity $\omega$, find the inclination of the rod to the horizon, and the tension of the string.

28. A cylinder rolls down a smooth inclined plane, the motion being checked by a string passing over the top and round a cylinder whose axis is fixed in the inclined plane, the string being attached to both. The diameter of the rolling cylinder is equal to the radius of the fixed, and both axes are horizontal. Find the velocity acquired in a given time.

29. An ellipse is dropped with its plane vertical and axis inclined to a horizontal rough plane; determine the velocity with which it leaves the plane supposed imperfectly elastic.

30. Two rough rods $A$, $B$ are placed parallel to each other and in the same horizontal plane. Another rough rod $C$ is laid across them at right angles, its centre of gravity being half-way between them. If $C$ be raised through any angle $\alpha$ and let fall, determine the conditions that it may oscillate, and shew that if its length be equal to twice the distance between $A$ and $B$, the angle $\theta$ through which it will rise in the $n^{\text{th}}$ oscillation is given by the equation

$$\sin\theta = \left(\frac{1}{7}\right)^n \sin\alpha.$$

31. A spherical hollow is made in a cube of glass, and a particle is placed within. The cube is then set in motion on a smooth horizontal plane so that the particle just gets round the sphere, remaining in contact with it. Find the velocity of projection.

32. Two equal uniform rods $AB$, $BC$, loosely jointed together at $B$, are laid on a smooth horizontal table, so that $\angle ABC = \alpha$. A blow is struck at $A$ perpendicular to $AB$; determine the direction and magnitude of the impulse at $B$, and shew that the initial motion of $A$ will be along $BA$, if

$$\tan\alpha = \frac{1}{\sqrt{2}}.$$

# HYDROSTATICS.

1. An iron spoon is gilded, and the mean specific gravity of the gilded spoon is 8, those of iron and gold are 7.8 and 19.4; find the ratio of the volumes and weights of the metals employed.

2. The specific gravities of gold and silver are 19.4 and 10.5; an article composed of gold and silver weighs 979 ounces in vacuo and 890 in water; compare the quantities of gold and silver in the article.

3. A vessel 8 inches high contains sea-water to a depth of 5 inches; olive-oil rests on the top of the water to the depth of one inch, and the remaining space is filled up with alcohol; find the pressure on a square inch of the base, having given that the specific gravities of sea-water, olive-oil, and alcohol, are 1.027, 0.915, 0.795, and that the weight of a cubic foot of distilled water is 1000 oz.

4. Equal volumes of oil and alcohol are poured into a circular tube so as to fill half the circle, shew that the common surface rests at a point whose angular distance from the lowest point is

$$\tan^{-1} \frac{4}{57}.$$

5. If in a circular tube two fluids be placed so as to occupy 90° each, prove that if the diameter joining the two open surfaces be inclined at 60° to the vertical, the densities are as

$$\sqrt{3} + 1 : \sqrt{3} - 1.$$

6. A circular tube contains three fluids, whose densities are as $1 : 2 : 3 + \sqrt{3}$ occupying area of 60°, 90°, 30°, respectively; find the position of the highest surface.

7. A fine tube is bent into the form of an equilateral triangle, and filled with three fluids which do not mix, whose specific gravities are $\rho$, $\sigma$, $\tau$, respectively, each occupying a side. The plane of the triangle is then made vertical, and one side is horizontal, shew that the opposite angle will be occupied by the lightest or the heaviest fluid, and that the sides will be divided in the ratio of the difference between its specific gravity, and that of the other two.

8. An isosceles triangle is placed in a fluid with its base in the surface, a rectangle is inscribed in the triangle, one side of which coincides with the base, and the opposite side is $\frac{2}{3}$ of the depth of the vertex, shew that the pressure on the rectangle is to the pressure on the triangle as 4 : 9. Also shew that this is the inscribed rectangle which has the greatest pressure.

9. A cycloid is placed in a fluid with its base in the surface and its plane inclined to the vertical. Its generating circle has its arc in the surface, and plane vertical. What is the inclination of the cycloid to the horizon, when the pressures on the cycloid and circle are the same? The distance of the centre of gravity of the cycloid from the base being seven-sixths of the radius.

10. A sector of a circle has its centre in the surface of a liquid and its bounding radii equally inclined to the surface, find the maximum pressure on the sector, when the perimeter of the triangle formed by the radii and chord is given.

11. An ellipse is placed with its major axis horizontal and the extremity of its minor axis in the surface of a liquid. Shew that the pressure on the circle inscribed in the triangle $SBH$ : pressure on the circle described on the minor axis as diameter ::

$$e^2 : (1 + e)^3.$$

12. Two equal parabolas are described passing through the corners of the square and touching at their vertices in its center, if the square be immersed in water, so that one side is in the surface of the water, and the axes of the parabolas horizontal, shew that the pressures on the parabolas are together equal to pressure on the remainder of the square.

13. An air-pump whose receiver and barrel are of volumes $A$, $B$ respectively, leaks in such a degree that, between every two strokes, a quantity of air enters the receiver equal to $\frac{1}{m}^{\text{th}}$ of the quantity at the end of a stroke, shew that after $n$ strokes the ratio of the density of air in the receiver to that of atmospheric air

$$=\left(1+\frac{1}{m}\right)^n \cdot \left(\frac{A}{A+B}\right)^n.$$

14. The receiver of an air-pump is 20 times that of the barrel, and a piece of bladder is placed over a hole in the top of the receiver, the bladder is able to bear 3 lbs. on the square inch, and the pressure of the atmosphere is 15 lbs., shew that the bladder will burst between the 4th and 5th strokes,

$$\log 2 = .30103,$$
$$\log 21 = 1.3222193.$$

15. In a vessel not quite full of water, and closed at the top by a flexible membrane, a small glass balloon, open at the lower part, contains sufficient air just to make it float, explain the principle upon which the balloon sinks, when the membrane is pressed in.

16. A cube floats in a liquid with one angle below the surface and three in the surface, shew that the specific gravity of the liquid is six times that of the cube.

17. Two equal and similar cones are joined together at their bases, and placed with their common axis vertical in a liquid, $\rho$, $\sigma$ are the specific gravities of the solid and liquid, shew that the distance of a vertex from the surface of the liquid is

$$h\sqrt[3]{\frac{(2\rho)}{\sigma}} \text{ or } h\sqrt[3]{\frac{\{2(\sigma-\rho)\}}{\sigma}}$$

as $\sigma >$ or $< 2\rho$, $h$ being the height of either cone.

18. Find the depth at which a hemisphere floats with its base downwards, when the specific gravities of the fluid and solid are as 16 : 11.

19. A paraboloid of revolution floats in water, shew that if a weight equal to eight times that of the paraboloid be placed in the paraboloid, it will sink to three times the original depth.

20. A cylindrical diving-bell, of height $a$, is sunk in water, until the water rises half-way up the axis; shew that the depth is

$$h - \frac{a}{2},$$

$h$ being the height of the water barometer.

21. A cylindrical tube enters a cylindrical vessel by an air-tight collar in the lid, and reaches very nearly to the bottom of the vessel. The depth of the vessel is equal to double the height of the barometer at the time, shew that if mercury be poured into the tube until it rises to the level of the lid, the vessel will be half filled.

22. A cylinder is filled with water to a height $2h$; find how much liquid, whose specific gravity is 2, must be poured into a second cylinder without weight, and whose base is one-fourth of the area of the first, so that the distance of the bases may be $h$, when it floats in the water. Shew that the quantity of liquid is one-twelfth of that of the water.

23. An isosceles triangular lamina is capable of turning round a hinge at the vertex, which is at a distance equal to $\frac{1}{4}$th of the base from the surface of a liquid, shew that the lamina will rest with its base vertical if the ratio of the specific gravity of the lamina to that of the liquid be as 5 : 32.

24. A triangular lamina $ABC$, right-angled at $C$, is attached to a string at $A$, and rests with the side $AC$ vertical and half its length immersed in fluid: shew that the density of the fluid : the density of the lamina :: 8 : 7.

25. A barometer has the area of the cistern 4 times that of the tube, and when the barometer stands at 30 inches the highest point of the tube is 2 inches above the upper surface; if a mass of air which at the density of the atmospheric air would fill one inch of the tube were admitted into the upper portion, shew that the column would be depressed 4 inches.

26. A barometer has its tube shortened, so that the mercury just reaches to the top, a bubble of air is introduced into the tube, which, under the atmospheric pressure, would occupy a length $c$ of the tube, shew that the mercury will descend in the tube a distance which is a mean proportional to $c$ and the height of the mercury in the common barometer.

27. A hemisphere is full of fluid, shew that if a plane be placed in the fluid which forms a circular section of the hemisphere, the ratio of the pressure when it is just immersed to the pressure when it is horizontal varies as the area.

28. A cylinder is filled with equal volumes of $n$ different fluids which do not mix; the density of the uppermost is $\rho$, of the next $2\rho$, and so on, that of the lowest being $n\rho$; shew that the whole pressures on the different portions of the curve surface of the cylinder are in the ratio

$$1^2 : 2^2 : 3^2 \ldots\ldots : n^2.$$

29. Two equal and uniform rods $AB$, $AC$ have a compass-joint at $A$ and are joined by a string at $B$, $C$; find the tension of the string when the system is floating in a liquid.

30. Spheres of different densities float in a fluid so that the area of the portion of the surface of each sphere, which is exposed to the fluid, is equal to the surface of a given sphere; shew that the heaviest sphere is that whose radius is the diameter of the given sphere, and whose density is $\frac{5}{32}$ of that of the fluid.

31. Find the center of pressure of a homogeneous liquid on a quadrantal area of a circle, whose center is in the surface and of which the bisecting radius is vertical.

32. Find the depth of the centre of pressure of the immersed portion of a Cardioid whose axis is in the surface.

33. A cylinder is placed with its axis vertical in a liquid whose density varies as the depth, and the density of the cylinder is the same as that of the liquid at a depth equal to half the height of the cylinder, find at what depth the cylinder will float.

34. If a hemispherical cup contain fluid whose density varies as the depth, the surface bisecting the axis of the cup, shew that the ratio of the whole pressure on the cup to the weight of fluid contained is $\frac{4}{3}$.

35. A cylinder floats in water with half of its axis which is vertical immersed. It is then put under the receiver of a condenser (filled with atmospheric air), so that the surface of the water is on a level with the mouth of the receiver. Find after how many strokes it will float with one-fourth of its axis immersed, having given $S.G$ of water $= 800 \times S.G$ of air, and vol of cylinder $= \frac{1}{2}$ vol of barrel $= \frac{2}{9}$ vol of receiver.

36. A semicircle is immersed with its base in the surface of a fluid, shew that if a rectangle be inscribed with one side in the base, the pressure on the rectangle is greatest, if a side subtends $60°$ at the lowest point of the semicircle.

37. A body when suspended from an extremity of an elastic string, whose natural length is $a$, stretches the string to a length $na$. If the body be now suspended from a point whose height above the surface of a fluid is $a$, and if the density of the fluid is $\frac{1}{n}^{\text{th}}$ that of the body, shew that it will sink to a depth

$$\frac{a}{n}(n-1)^2.$$

38. A semi-cycloid is placed with its right angle downwards in a fluid. Find the position of equilibrium, the coordinates of the centre of gravity referred to the right angle being $\frac{5a}{6}$ and $a\left(\frac{\pi}{2} - \frac{8}{9\pi}\right)$; where $(a)$ is the radius of the generating circle. Shew that it cannot rest with either extremity of its arc in the fluid.

39. $ABC$ is a right-angled triangular plate, and it floats with its plane vertical and the right angle $C$ immersed in water; prove that if its specific gravity be to that of water as $2:5$ and $CB:CA=5:4$, $CB$ is cut by the surface of the water at a distance from $C$ equal to $CA$.

Find the metacentre of the floating plate.

40. A cone of specific gravity $\rho$ floats in a fluid of specific gravity $\sigma$, $2\theta$ being the vertical angle. Shew that the depth of the vertex, supposed downwards, $=\left(\frac{\rho}{\sigma}\right)^{\frac{1}{3}}\times$ height of cone. Shew also that the cone can float in an inclined position if

$$\cos\theta>\left(\frac{\rho}{\sigma}\right)^{\frac{1}{6}},$$

Find the condition that the upright position may be stable.

41. An isosceles triangular lamina is suspended from the vertex over a vessel, find what quantity of fluid must be poured in before the equilibrium of the lamina becomes unstable.

42. If in Prob. 17, $\sigma>2\rho$, shew that the equilibrium cannot be stable, unless the vertical angle of the cones be greater than $60°$.

43. A paraboloid is placed in a fixed hemispherical cup, and filled with fluid up to the focus; shew that the equilibrium will be indifferent if the radius of the cup is double the latus rectum.

44. Shew that if $l$, $l'$ be the latera recta of the principal sections of an elliptic paraboloid cut perpendicular to the axis, and $h$ the height; when it floats in the water, the equilibrium is stable with regard to one section, and unstable with regard to the other, if the specific gravity lie between

$$\left(h-\frac{3l}{4}\right)^2, \text{ and } \left(h-\frac{3l'}{4}\right)^2.$$

45. A hemisphere floats in a liquid with its flat side downwards, prove that the distances from the base of the center of gravity and metacentre are in the ratio of the densities of the fluid and the solid. Prove that the same relation holds when the displacement is through a finite angle.

46. If a parabolic lamina of height $h$ will rest in an inclined position with its plane vertical in a fluid, and the focus in the surface, and $c$ be the length of the line of floatation, prove that

$$2c = 3h - 5a,$$

where $4a$ is the latus rectum.

47. A cylinder shut at one end, on which it stands vertically, is closed at the other by an air-tight piston, which is of weight $W$, and area $\kappa$, find to what depth the piston sinks; and, if $E$ be the expansion for $1^\circ$ of temperature, shew that the piston will be forced out if the temperature of the contained air be increased beyond $\dfrac{W}{\kappa E\Pi}$ degrees.

48. A cylinder is placed with its axis horizontal, and a piston separates two gases whose pressure keeps it in equilibrium in the middle point, if the two gases be unequally heated, prove that the piston will rest at a distance equal to

$$xe\,(t \backsim t'),$$

$t$, $t'$ being the increased temperatures, $2a$ the length of the axis, and $e^2$ being neglected.

49. If in Prob. 20, the temperature of the air in the cylinder be raised $t^\circ$, shew that the water recedes a distance nearly equal to

$$\frac{2ahet}{2h+a}.$$

50. If in a steam-engine the steam be cut off when the piston has advanced over $\dfrac{1}{n}^{\text{th}}$ of the cylinder, find the work done during one stroke of the piston, supposing the pressure of steam in the boiler to be $m$ times that of the atmosphere.

51. A Hawksbee's air-pump is forced with a single barrel, find the work done in the $n^{\text{th}}$ stroke of the piston.

52. A circular tube is half-full of fluid, and is made to revolve uniformly round a vertical tangent line, with angular velocity $\omega$, if $(a)$ be the radius, prove that the diameter passing through the open surfaces of the fluid is inclined to the horizon, at an angle

$$\tan^{-1}\frac{\omega^2 a}{g}.$$

53. A vertical cylinder containing water rotates uniformly about its axis. A paraboloid of revolution, whose specific gravity is .5, just floats to the water's edge, with its vertex downwards. Shew that the angular velocity is $\sqrt{\frac{g}{l}}$, $l$ being the latus rectum.

54. Equal quantities of two liquids of different specific gravities are contained in a cylinder which rotates with uniform angular velocity about its vertical axis, and exactly fill it; find the position of the common surface.

55. In the axis of a cylindrical vessel revolving round its axis is placed a center of force, which attracts with a force which varies as the distance. Find the shape of the surfaces of the fluid, supposing different angular velocities communicated.

56. Find the form of the surfaces of equal pressure when a fluid is in relative equilibrium, under the action of forces which vary as the distance and are equal at equal distances and tend to the angular points of a triangle, about a line through the center of gravity of which, and perpendicular to its plane, the fluid is revolving with uniform angular velocity.

57. A continuous mass of an incompressible fluid extends to an infinite distance in a vacuum; if a spherical portion be annihilated and every point of the surface receive an equal velocity outwards in the normal, find the time of swelling to four times the size, and the pressure at any point of the interior.

# OPTICS.

1. A MAN stands at the edge of a brook, on the opposite side of which, 50 feet from the opposite edge, stands a tower 30 feet high, whose top is just visible by reflection; find the breadth of the brook, the man's eye being 6 feet above the level of the water.

2. $AB$ is a straight line, along which a bright point moves uniformly; shew that the shadow of a fixed opaque point upon a plane parallel to $AB$ moves in a straight line; and find its velocity.

3. If two plane mirrors be inclined at an angle of 40°, shew that there are exactly 8 images of a luminous point placed in the plane bisecting the angle between them.

4. A luminous point is placed equidistant from two plane mirrors inclined at an angle 18°; find the number of images formed.

5. Three plane reflectors are placed so that their section by a plane perpendicular to each is a triangle whose angles at the base are each double of the angle at the vertex, prove that there are fourteen images of a luminous point placed at the centre of the inscribed circle.

6. A luminous point is placed between two plane mirrors inclined to each other at a given angle: shew that if the angle between the mirrors be $\frac{\pi}{n}$, where $n$ is an integer, the number of images will be independent of the position of the luminous point.

7. A luminous point is placed in the focus of a paraboloid, shew that the illumination at any point $P$

$$\propto (SP)^{-\frac{5}{2}}.$$

8. A luminous point is placed immediately above the center of an annulus whose internal and external circumferences are distant $a$ and $b$ from the luminous point, shew that the illumination of the annulus varies as

$$\frac{1}{a} - \frac{1}{b}.$$

9. A thin flat ring is illuminated by a luminous point, directly above its center, shew that the illumination of the whole varies as

$$\sin\theta\cos^2\theta,$$

where $\theta$ is the angle subtended by the radius at the luminous point. What must be the height of the point in order that the illumination may be the greatest?

10. A luminous point is placed at the bottom of a hemispherical cup; shew that the illumination at any point of the cup varies inversely as the distance from the luminous point.

Find the whole illumination of the cup.

11. A luminous point is placed in the focus of an ellipse, find the illumination at any point, and shew that it will have no minimum value unless the eccentricity is = or > $\frac{2}{3}$.

12. A luminous point is placed in the circumference of a circle, shew that if $\alpha$ be the illumination of a point at the extremity of the diameter through the luminous point, and $\alpha_1$, $\alpha_2$ those of points at the extremities of any other diameter, then

$$\frac{1}{\alpha_1^{\,2}} + \frac{1}{\alpha_2^{\,2}} = \frac{1}{\alpha^2}.$$

13. Prove that, if a hemispherical surface be self-luminous throughout the interior, and of uniform brightness, the portion within the surface, of a plane upon which it is placed, with its base in contact, will be uniformly illuminated.

14. A small plane area is placed at the focus $S$ and perpendicular to the axis of a paraboloid which is uniformly luminous; if the axis of the paraboloid be $\frac{3}{4}$ths of its latus rectum, shew that the illuminations of the two sides of the area are as 1 : 4.

15. A uniformly luminous paraboloid touches a plane, shew that the illumination of the point on the plane at which the axis meets it varies inversely as the cube of the distance of the focus from the plane.

16. A cylinder of fluid revolves uniformly about its vertical axis, and a small object at the bottom is viewed by an eye above the fluid. Prove that in all positions the eye sees a distinct, erect, and diminished image of the object.

17. Light proceeding from a luminous point is incident on one of the faces of a prism, whose section is an equilateral triangle, is internally reflected at one of the other faces and emerges at the third. Trace the pencil by which an eye situated anywhere in the plane which passes through the luminous point and is perpendicular to the faces of the prism, will see the image of the point.

18. The vertical angle of an isosceles prism is $2\alpha$; $D$ is the deviation of a ray perpendicular to the base, internally reflected, and emergent. If there be one reflection

$$\cos(\alpha + D) = \mu \cos 3\alpha,$$

if there be two reflections

$$\cos(\alpha + D) = \mu \cos 5\alpha.$$

19. A ray of light is incident at an angle $\phi$ on a side of an isosceles prism, in a plane perpendicular to the edge of the prism, and after internal reflection at the base emerges from the opposite side. Prove that the deviation is

$$2\phi + \alpha,$$

$\alpha$ being the vertical angle of the prism.

Hence, explain why the image of an object by rays reflected internally is not fringed with colour as in direct refraction.

20. A ray of light is incident perpendicular to the side of an isosceles prism, whose vertical angle is 30°, in a plane perpendicular to its edge; and after reflection at the opposite side and base reaches the point of incidence, shew that the length of its path in the prism is equal to one of the sides.

21. A plate of glass is terminated by a slant side, so that if $AB$, $CD$ be parallel sections perpendicular to the faces of the plate, $AC$ is inclined at an angle $\alpha$ to $AB$.

Shew that, in order that a ray refracted at $AB$, and reflected internally at $AC$, may emerge at $CD$, the angle of refraction at $AB$ must be between

$$2\alpha - \pi + \sin^{-1}\frac{1}{\mu} \text{ and } 2\alpha - \pi - \sin^{-1}\frac{1}{\mu},$$

if $\alpha > \frac{\pi}{2}$.

22. If the directions of emergence and incidence in Prob. 22 be perpendicular, prove that

$$\sin 2\phi = \frac{\mu^2 \sin^2 2\alpha - 1}{\cos 2\alpha},$$

$\phi$ being the angle of incidence.

23. A luminous point is placed half-way between the centre and surface of a refracting sphere; shew that if the index of refraction out of the surrounding medium into the sphere be less than 2, every ray will emerge, but that if it be greater than 2, every ray incident on a certain belt of the surface will be totally reflected.

24. A hemisphere and its base have their internal surfaces polished, and a luminous point is placed in the bisection of the radius perpendicular to the base, shew that real and virtual images will be formed at points whose distances from the base are in harmonic progression.

25. A number of circular laminæ of different radii are placed in order on a common axis. They are at finite distances from one another such that each subtends a greater angle than its predecessor at a luminous point in front of the least lamina. Shew that if any of the laminæ except the greatest be removed the total illumination remains the same.

26. A pencil of parallel rays is incident upon a refracting sphere. Shew that the vertical angle of the cone in which those rays emerge which pass through the extremity of the diameter to which the incident rays are parallel, is

$$4 \cos^{-1} \frac{\mu}{2}.$$

27. A small coin is placed at the bottom of a hemispherical cup filled with liquid whose refractive index is $\sqrt{\frac{3}{2}}$; find the height of the eye above the plane of the rim of the cup at a given distance from it, in order that the coin may be just visible; and find the depth of the primary focus in this position.

28. $C$ is the center of a refracting sphere, $Q$ is the origin of a small pencil refracted at $R$ so that the refracted pencil has its axis parallel to $QC$, shew that the distance of the secondary focus from $R$ is $\frac{1}{2} QC$.

29. A refracting substance, whose index of refraction is $\mu$, is terminated by a spherical surface $BAC$, center $O$, and a plane surface $FDE$ perpendicular to the axis $AOD$ of $BAC$. A small direct pencil diverging from $Q$, in $AD$ produced, is refracted at $ED$, reflected internally, at $BAC$, and emerges at $ED$, and then diverges from the same point as if reflected at $ED$, shew that

$$\mu^2 DQ^2 = AD \,.\, OD.$$

30. Find the primary focus of a small pencil of rays diverging from a point in the circumference of a hollow reflecting sphere after two internal reflections.

Shew that after a very large number of such reflections the primary foci all lie in a circle concentric with the sphere.

31. A small pencil of parallel rays is incident on a refracting sphere, at such a point that the axis of the refracted pencil passes through the extremity of the diameter parallel to the incident pencil; shew that the distances of the primary and secondary foci from the point of incidence are as

$$\mu^2 : 4.$$

32. A concave mirror and convex lens of equal focal length $f$ have a common axis $AC=f$, at the extremities of which they are placed. $Q$ is an origin of light on $AC$, and $q$ is the image after reflection at the mirror and refraction through the lens $C$, shew that

$$AQ.Cq=f^2.$$

33. When the origin of light is 6 inches from a convex spherical reflector of one foot radius and 4 inches aperture, find the radius and position of the least circle of aberration.

34. Find the geometrical focus of a small pencil of rays refracted through a double concave lens, the radii of which are 9 and 18 inches, the refractive index $\frac{3}{2}$, and the distance of the origin of light 12 inches.

35. Two convex lenses of focal lengths 1 inch and 2 inches, have a common axis, and are at a distance 3 inches; if a small object be placed at a distance of 4 inches from the first lens, shew that the image will be seen at a distance 10 inches, and is double the size of the object placed at that distance; trace the course of a pencil to the eye, from a point not in the axis.

36. A luminous point is placed at a distance $u$ from the center of a convex lens, upon the axis; shew that if a plane mirror can be placed behind the lens so as to make the rays incident upon it converge to the luminous point after reflection, the distance of the radiant from the lens must be between $f$ and $2f$, $f$ being the focal length of the lens.

If the distance of the radiant be $\frac{3f}{2}$, that of the plane mirror must be $\frac{3f}{4}$.

37. A pencil of rays is incident parallel to the axis of a concavo-plane lens, and is reflected internally and afterwards emerges, shew that the rays diverge from a distance

$$\frac{r}{2(\mu-1)}.$$

38. If a small pencil be obliquely and centrically refracted through a thin lens whose breadth is $2a$, and $x$ be the distance of the circle of least confusion from the centre, and $\rho$ its radius, prove that

$$\frac{1}{x}=\frac{1}{v_1}+\frac{1}{v_2}=\frac{a}{\rho}\left(\frac{1}{v_1}-\frac{1}{v_2}\right).$$

39. A pencil of parallel rays is incident upon a lens, shew that the distance of the principal focus from the centre of the first surface is a maximum or minimum, when the ratio of the radii is

$$(\mu-1)^{\frac{1}{2}} : 1 \mp (\mu-1)^{\frac{1}{2}},$$

the radius of the second surface being given.

40. Two equal and similar concavo-convex lenses of small thickness are placed so as to contain in the hollow part between them a fluid, find the geometrical focus of a pencil of parallel rays after refraction through them.

41. An astronomical telescope whose object-glass has a focal length $18f$, is adjusted with an eye-piece whose two convex lenses have equal focal lengths $f$, and are at a distance $3f$; shew that the magnifying power is 18, and that the distance between the object and field-glasses is $20f$.

42. Shew that an astronomical telescope furnished with a Ramsden's eye-piece, the focal length of the object-glass of which is 60 times that of either lens of the eye-piece, magnifies 80 times.

43. The object-glass of an astronomical telescope is composed of two lenses in contact, one convex, the other concave, whose focal lengths are $10f$ and $110f$ respectively, and a Ramsden's eye-piece, whose lenses are each of focal lengths $f$ and separated by a distance $\frac{2f}{3}$; find the distance of the eye-piece from the object-glass when a star is seen distinctly by an ordinary eye.

44. In the common astronomical telescope, without a stop, if the focal lengths of the object-glass and eye-glass be 10 feet and 6 inches respectively, and the breadth of the object-glass 5 inches, shew that the angular breadth of the portion of the field of view visible by imperfect pencils is 13′ 18″ nearly.

45. The focal lengths of the object-glass and eye-glass of an astronomical telescope are as 54 : 1. A convex lens of twice the aperture and focal length of the eye-glass is placed between them at a distance from the eye-glass equal to twice its focal length. Find the effect upon the magnifying power and field of view, the instrument being adjusted for distinct vision.

46. The focal lengths of the mirrors in a Gregorian telescope are 32 and 4 inches, and the distance between $F$ and $f$ is $\frac{1}{2}$ of an inch. Find the focal length of each of the lenses of a Ramsden's eye-piece, in order that when adjusted to distant objects, the center of the field glass may coincide with that of the surface of the large mirror.

47. The dispersive powers of two media for two of the most vivid colours being .03 and .05, find what must be the focal lengths of two lenses, which in contact form the object-glass of a telescope of 30-inch focal length.

48. $P$ is any point in a parabola, $Q$ is the image of a luminous point at the focus formed in a plane mirror coinciding with the normal at $P$; shew that the locus of $Q$ is a parabola of half the dimensions of the given parabola.

49. A luminous point moves along the directrix of a reflecting parabolic arc, shew that the distance of the geometrical focus from the arc is equal to one-fourth of the radius of curvature.

50. $AP$ is a parabola, $A$ the vertex, a bright point moves in the arc of the parabola, and at any point $P$ an image is formed by a concave lens placed at the focus $S$, whose focal length is $AS$; shew that the locus of the image is an ellipse whose eccentricity is $\frac{1}{3}$ and major axis

$$\frac{3AS}{2}.$$

51. Explain why the horizontal base of a reservoir of water seems to slope upwards in all directions from the point nearest the observer.

52. Give a geometrical construction for the caustic which corresponds to a bright point placed below the surface of water whose refractive index is $\frac{4}{3}$, and employ the caustic to discover the position in which a fish at a given depth below the surface, would be seen by an eye placed so as to see by very oblique pencils.

53. A luminous point is placed at the focus of an ellipse polished in the interior, and a convex lens at the center, whose axis is that of the ellipse, shew that in order that after refraction through the lens and reflection at the curve the light may converge to the center, the lens must have a focal length which is to the major axis as

$$e^2 : (1+e)^2.$$

54. An ellipse has eccentricity $\frac{2}{3}$, shew that if a bright point in the foot of the normal at any point converge, after direct reflection at the curve, to the extremity of the diameter of curvature at that point, the inclination of the normal to the axis is 60°.

55. A spherical glass shell, $\mu = \frac{3}{2}$, whose external radius is double its internal radius, is placed with its centre in the axis of a paraboloid, midway between the vertex and focus. If a small pencil of rays, incident on the shell parallel to the axis, pass through the focus, after refraction through it, prove that the external radius equals $\frac{1}{12}$th of the latus rectum.

56. At the centre of an ellipse is placed a hollow spherical glass shell, whose external radius is two thirds of the distance between the focus and nearer vertex. If the radii of the shell are in the ratio $e : 1$, a pencil of rays incident parallel to the axis will have the focus of the ellipse for geometrical focus.

57. An object is placed half-way between two parallel plane mirrors, prove that the brightness of the successive images are as the squares of numbers in harmonic progression.

58. Three candles, of equal height, are placed on a table, at equal distances from one another. Half-way between each and the ceiling hangs a ball. Shew that the areas of the hexagon and triangle, in the angular points of which the nine shadows fall, are in the ratio of 13 : 1.

59. A rectangular glass vessel is filled with water and light is admitted in all directions through a small hole in the lid, shew that the boundaries of the light on the sides are parabolas.

60. A uniformly luminous sphere has its center in the surface of a hollow sphere, prove that the illumination at any point of the hollow sphere varies inversely as its distance from the center of the luminous sphere, a portion being cut off from the hollow sphere so that the view of it is not impeded.

61. Two luminous spheroids are formed by the revolution of an ellipse, semi-axes $(a, b)$, respectively about its major and minor axes. If $I$ be the illumination, by the first spheroid of a point on its axis, at a distance $(a)$ from the vertex; $I'$ the illumination by the second of a point at a distance $(b)$ from its vertex. Prove that

$$\frac{I}{I'} = \frac{b^2}{a^2} \cdot \frac{a^2 + 3b^2}{b^2 + 3a^2}.$$

62. The angular distance of a point from the centre of the field of an astronomical telescope is $\alpha$, and the simple object-glass and eye-glass of the telescope have the same dispersive power $\varpi$, and focal lengths $F$ and $f$; shew that the angular breadth of the spectrum is

$$\varpi\alpha\left(1 + \frac{F}{f}\right).$$

63. When a ray of light is reflected at two plane surfaces inclined at an angle $\alpha$ to one another, so that the angles of incidence at each surface are the same, and the planes of reflection perpendicular to each other, prove that

$$\cot^2\theta = \cos\alpha,$$

if $\theta$ be the angle between the plane of first reflection and the plane perpendicular to the line of intersection of the reflectors.

64. A ray of light is reflected at two plane mirrors, the angles of incidence on each being 45°, and the planes of reflection at right angles. Shew that the inclination of the

mirrors is 60°, the deviation 90°, and the angle between the plane of first reflection and that perpendicular to the intersection of the mirrors is

$$\tan^{-1}\sqrt{2}.$$

Prove that these results are true, if the mirrors instead of the planes of reflection are at right angles, both the angles of incidence being 60°.

65. A hollow cylinder, whose height ($2h$) is to the diameter of its base as 4 : 3, is open at the top, and has its interior surface polished. It is partially filled with glass, $\mu = \frac{3}{2}$. A ray of light, passing through one extremity of a diameter of the top, is reflected at the middle point of the opposite side of the cylinder, refracted by the glass, reflected at the centre of the base, and then by a similar course passes out, at the other extremity of the diameter. Shew that the depth of the glass in the cylinder is

$$\frac{9h}{10}.$$

66. A luminous point is placed in the center of gravity of a perpendicular section of a triangular prism, shew that the areas of the section formed by lines drawn from the geometrical foci to the edges and the sides from which the rays emerge are all equal.

Prove that the triangle formed by joining the three geometrical foci : triangular section of the prism

$$\overline{\mu - 1}|^2 (\sin^2 A + \sin^2 B + \sin^2 C) : 9\mu^2.$$

67. Two prisms of different refractive indices ($\mu$, $\mu'$) are placed with two faces in contact and their edges coincident, shew that no ray will pass through the combination if the sum of their refracting angles be

$$> \sin^{-1}\frac{1}{\mu} + \sin^{-1}\frac{1}{\mu'}.$$

68. If a refracting substance have a surface formed by the revolution of a cardioid about its axis, and a luminous point

be placed in the focus; prove that rays will emerge parallel to the axis which are inclined to the axis at an angle

$$\cos^{-1}\frac{\mu-1}{2}.$$

69. A luminous lemniscate $r^2=a^2\cos 2\theta$ is immersed in fluid, $\mu=\sqrt{2}$, with its plane vertical and origin in the surface. Shew that the equation to the image, seen by an eye placed close to the origin, is

$$r=\frac{a}{2\sqrt{2}}\left(3\cos\theta-\frac{2}{\sec\theta+\cos\theta}\right).$$

70. A sphere is formed of substance of variable refractive power and a ray of light is incident so as to enter at a given angle. If the index of refraction from air into the substance at any distance from the centre be $\mu$, shew that the path of light in the sphere is a rectangular hyperbola if $\mu \propto$ as distance, and a circle passing through the centre if $\mu \propto$ (distance)$^{-2}$.

71. If a refracting medium be such that the refracting index $\propto \frac{1}{r^n}$, $r$ being the distance from a fixed point, prove that the path of a ray of light in the medium is given by an equation of the form

$$r^{n-1}=c^{n-1}\cos(n-1)\theta.$$

What is the form if $n=1$?

72. $\alpha$, $\beta$ are the vertical angles, in the plane of $xz$, of the luminous cones which produce respectively interior and exterior conical refraction. Shew that

$$ac\tan\alpha+b^2\tan\beta=0,$$

$a$, $b$, $c$ being the optical constants.

# SPHERICAL TRIGONOMETRY AND ASTRONOMY.

1. If in a spherical triangle right-angled at $A$, the angle $B$ be double of $C$, shew that

$$(1)\quad \operatorname{cosec} C = 2\cos c = 2\cos\frac{a}{2}.$$

2. Shew that in an equilateral spherical triangle

$$2\cos A = 1 - \tan^2\frac{a}{2}.$$

3. If $c$ be the hypothenuse of a right-angled spherical triangle, prove that

$$\sin^2\frac{c}{2} = \sin^2\frac{a}{2}.\cos^2\frac{b}{2} + \cos^2\frac{a}{2}.\sin^2\frac{b}{2}.$$

Deduce the corresponding result when the triangle is plane.

4. $ABC$ is an isosceles spherical triangle in which each of the equal sides is double of the third $BC$, prove that

$$\operatorname{cosec}\frac{A}{2} = 4\cos BC.\cos\frac{BC}{2}.$$

5. $ABC$ is a spherical triangle, and $AD$, $BE$, $CF$ are three arcs of great circles intersecting in $O$ and perpendicular to the sides $BC$, $AC$, $AB$ respectively, shew that

$$\sin OD.\tan BC = \sin OE \tan AC = \sin OF.\tan AB.$$

6. $ABC$ is a spherical triangle, bisect $BC$ in $D$ and $AD$ in $E$, and shew that

$$\cos CE + \cos BE = 2 \cos BD \,.\, \cos DE.$$

Test the result by Napier's rules when $AB = AC$. Deduce the corresponding result if the triangle be plane.

7. $ABC$, $ADC$ are two great circles whose planes are at right angles, $BD$ is a great circle bisecting the contained line, shew that

$$-\cos BD = \cos^2 AB = \cot^2 ADB.$$

8. $ABC$ is a spherical triangle in which the side $AB$ is a quadrant, and through $A$, $B$ arcs of great circles are drawn bisecting $BC$, $AC$ in $D$ and $E$, shew that

$$\cos^2 AC \sec^2 AD - \cos^2 BC \,.\, \sec^2 BE = 2\{\cos BC - \cos AC\}.$$

9. $ABC$, $ABD$ are two spherical triangles, such that $AC + CB = AD + DB$; if $O$ be the middle point of $AB$, shew that

$$\frac{\cos CO}{\cos DO} = \frac{\cos \dfrac{BC - AC}{2}}{\cos \dfrac{AD - BD}{2}}.$$

10. $ABC$ is a spherical triangle, and $AD$, $BD$ are arcs of great circles perpendicular to $AC$ and $BC$, shew that

$$(1)\quad \tan \frac{C - D}{2} \,.\, \tan \frac{C + D}{2} = \cos^2 \frac{c}{2} \,.\, \frac{\cos (A + B)}{\cos (A - B)}.$$

$$(2)\quad \frac{\tan AD}{\cos B \,.\, \cos b} = \frac{\tan BD}{\cos A \cos a} = \frac{\sin c}{\cos CD \sin D}.$$

11. If in the last Problem a great circle be drawn through $C$ making an angle $\theta$ with $CD$, shew that

$$\tan \theta = \cot \frac{C}{2} \,.\, \frac{\sin (a \sim b)}{\sin (a + b)}.$$

12. Prove that in any spherical triangle

(1) $$\cot a = \frac{\sin 2c \,.\, \cos B - \sin 2b \,.\, \cos C}{\cos 2b - \cos 2c}.$$

(2) $$\cos B + \cos C = \frac{2 \,.\, \sin (2s-a) \,.\, \sin (s-b) \sin (s-c)}{\sin a \,.\, \sin b \,.\, \sin c}.$$

In (1) deduce the corresponding result when the triangle is plane.

13. From the area of a spherical polygon deduce the surface of a portion of a sphere, cut off by a plane at a given distance from the centre.

14. If $r_1, r_2, r_3$ be the radii of the circles touching one side of a spherical triangle, and the other two sides produced and $r$ the radius of the inscribed circle, then $\tan r_1$, $\tan r_2$, $\tan r_3$ are given by the equation

$$z^3 - P_2 \cot r \,.\, z^2 + P_1 \sin s \,.\, z - \sin^2 s \,.\, \tan r = 0,$$

where $$P_2 = \sin (s-a) \sin (s-b) + \sin (s-c) \sin (s-b) + \sin (s-a) \sin (s-c),$$

$$P_1 = \sin (s-a) + \sin (s-b) + \sin (s-c).$$

15. The Earth's radius being 4200 miles nearly, find the dip of the horizon in seconds to a person 12 feet above the level of the sea.

16. Supposing the angular motions of Jupiter and Saturn round the sun to be uniform and in the ratio of 5 : 2; find the angular distance between the two positions of conjunction. If the period of Jupiter be 12 years, what is the time from conjunction to conjunction, and the time between two conjunctions in the same part of their orbits?

17. Find the distance of a body whose parallax is $100''$.

18. The Moon's longitude at noon on May 30th is $96°, 35', 14''$; will the Moon be visible before sunrise or after sunset?

19. If the Moon be 18 days old, find approximately the direction and form in which she will be seen at midnight in the middle of summer.

20. Shew that two places in latitude 45°, and whose difference of longitude is 90°, are $\frac{2}{3}$ of the distance of places on the equator with the same difference of longitude.

21. The Moon's angular distance from Spica Virginis on May 8th at Greenwich was 24°, 10′, 42″ at 9 p.m., and 22°, 33′, 52″ at midnight, the distance at a certain place cleared of refraction and parallax was 23° at 8 p.m.; what is the longitude of the place?

22. In determining the latitude by means of the mural circle and a known star, shew that the error caused by a deviation error $D$ is

$$\tfrac{1}{2}\cos l \sin z \sec\delta . D^2.$$

23. $\alpha$ Pegasi has $R.A.$ 22h, 57m on July 20th 1851, and $N.D.$ 14° 24′. Find during what part of the night the star will be visible, and in what quarters of the heavens it will be first and last seen.

24. If the Moon's orbit be inclined at an angle $\omega$ to the equator, shew that when the Moon rises at the same sidereal time on two successive nights, the latitude is $90° - \omega$.

25. A pole is placed in latitude 15°, and at noon no shadow is cast, find the length of the pole in order that at 10 o'clock in the morning the shadow may be 7 feet long.

At what time of the year will this happen?

26. Find the change in azimuth of the point of the Sun's rising for a given change of declination.

27. If the North and South Declination of two stars be equal, which are on the horizon at the same time, shew that their respective distances from the North and South points of the horizon are equal. Shew that when the star with South Declination sets the other star has an hour angle

$$2\pi \backsim 3h$$

where $h$ was its hour angle at rising.

28. If the horizontal angle between two stars in the equator be observed by a Theodolite at different times in a place of 45° latitude, and $\theta$, $\phi$ be the greatest and least of these angles, shew that

$$\tan\frac{\theta}{2} = 2\tan\frac{\phi}{2}.$$

29. The hour angles of a star when rising above the horizon, and when its apparent motion is vertical, are $h$, $h'$, shew that if $l$, $\delta$ be the latitude of the place and the declination of the star

$$\tan^2 l + \cos h \cos h' = 0,$$
$$\tan^2 \delta + \cos h \sec h' = 0.$$

30. If $a$ be the $R.A.$ of a star in the equator, whose altitude is equal to the latitude $l$ at the sidereal time $s$, shew that

$$\cos(a - s) = \tan l.$$

Also if $A$ be the azimuth reckoned from the south point

$$\cos A = \tan^2 l.$$

31. If a star has declination $\delta$ and $R.A.$ zero, shew that the cotangent of the angle which the direction of aberration makes with the equinoctial colure is equal to

$$\pm(\tan\odot \sin\delta \sec\omega + \cos\delta \tan\omega,)$$

and if the longitude of the Star be that of the Sun, this expression becomes

$$\operatorname{cosec}\lambda \tan\odot,$$

where $\lambda$ is the latitude of the star.

32. A meteor is observed by each of two observers, one of whom is due north of the other at a known distance, to fall due north from one known star to another, shew how the height of the meteor and the inclination of its path to the horizon may be determined.

# ANSWERS TO QUESTIONS.

## ALGEBRA.

2. (1) $\dfrac{ax + by}{ax - by}$. (2) $\dfrac{xe^y + ye^x}{xe^x + ye^y}$.

9. (1) $x = -2, \ (\pm\sqrt{7} + 2)$.

(2) $x = \pm\sqrt{2}, \ \pm\sqrt{(2 + 2^{\frac{2}{3}})}$.

(3) $x = 1, \ -1, \ -3, \ -5$.

(4) $x = \dfrac{\pm 4\sqrt{(39)} - 57}{100}$.

(5) $\begin{cases} x = \frac{1}{2}, & y = 3, \ \frac{1}{3}, \\ x = -\frac{1}{2}, & y = -3, \ -\frac{1}{3}. \end{cases}$

(6) $\begin{cases} x = \pm 1, & y = \pm 2, \\ x = \pm 2, & y = \pm 1. \end{cases}$

(7) $x = b, \ y = a$. (8) $x = 1, \ y = 4, \ z = 2$.

(9) $\begin{cases} x = 16, & y = 4, \\ x = 6 \pm \sqrt{(117)}, & y = -6 \pm \sqrt{(117)}. \end{cases}$

(10) $\begin{cases} x = 0, & y = 0, \\ x = 26, & y = 130, \\ x = 9\frac{1}{3}, & y = -37\frac{1}{3}. \end{cases}$

(11) $\begin{cases} x = b(ab - 1), \ y = a(ab - 1), \\ x = \dfrac{b}{2}\left\{\pm\sqrt{\{(1 + ab)(1 - 3ab)\}} - (1 + ab)\right\}, \end{cases}$

$$y = \frac{a}{2}\left\{\mp\sqrt{\{(1 + ab)(1 - 3ab)\}} - (1 + ab)\right\}.$$

(12) $\begin{cases} x=0, \ y=0, \\ x=-\dfrac{1375}{81}, \ y=\dfrac{50}{9}. \end{cases}$

(13) $x=0$, $x=\left\{\dfrac{\sqrt{(5a^2+4)}+3a}{2a(1-a^2)}\right\}^2$,

$x=\left(\dfrac{1}{a^2-1}\right)^2$, and $x=\left\{\dfrac{\sqrt{(5a^2+4)}-3a}{2a(1-a^2)}\right\}^2$, satisfy the equation if the negative value of the second radical be taken in each case.

(14) $\begin{cases} x=\dfrac{15\sqrt{(221)}\pm 221}{2}, \\ y=\pm\sqrt{(221)}. \end{cases}$

10. $15\frac{1}{2}$ miles. 11. £12., £8. 12. 14h. 46m.

13. $4\frac{1}{2}$ miles. 14. The second. 15. $3\frac{1}{3}$ days.

16. $A$ takes $\dfrac{p(p+r)-q(q+r)}{p-q-r}$,

$B$ takes $\dfrac{p(p+r)-q(q+r)}{p+r-q}$.

17. The watch gains 5 hours in 24.

19. $x=\pm 1, \ -\dfrac{b}{2a}\pm\dfrac{\sqrt{(b^2-4a^2)}}{2a}$.

32. 380. 33. 28561. 34. 6720. 35. 4845.

36. 27760, 280. 47. $x=\dfrac{1}{a}$, $y=\dfrac{1}{c}$.

50. (1) $\begin{cases} x=4, \ 6, \ 12, \\ y=16, \ 12, \ 16. \end{cases}$ (2) $\begin{matrix} x=2, \ 3, \ 5, \\ y=7, \ 3, \ 1. \end{matrix}$

51. 454 yards. 53. $\left(\dfrac{a}{b}\right)^{\frac{1}{12}}+\left(\dfrac{b}{a}\right)^{\frac{1}{12}}=m^{\frac{1}{9}}$.

57. If $m$ and $n$ be the number of teeth in each wheel, the number required will be given by the equation $mx-ny=\pm 2$.

59. $\dfrac{1}{216}$. 60. $\dfrac{2}{7}$, $\dfrac{11}{21}$. 61. 1728.

62. He ought to take 19 to 9.

## THEORY OF EQUATIONS.

1. $x^4 - 2x^3 - 9x^2 + 10x - 2 = 0$.

2. (1) $\pm\sqrt{(-3)}$. (2) $\pm\sqrt{3}$. (3) 3, 2.

3. $p_1 - \dfrac{p_{n-1}(p_1^2 - 2p_2)}{p_n}$. 6. $x = 2, 2, 3, \dfrac{3}{2}$.

7. $y^3 - 2250y + 1575 = 0$, where $y = 225x$.

8. (1) $x = 3, \dfrac{1}{3}, -2, -\dfrac{1}{2}$.

(2) $x = 1, 1, 3, \dfrac{1}{3}, \dfrac{\pm\sqrt{(-7)} - 3}{4}$.

9. $ry^3 - (r+1)qy^2 - (r+1)^3 = 0$. 10. $5, \dfrac{500}{603}$.

11. (1) $\dfrac{19}{9}, \dfrac{5}{8}$ of the positive roots; $-8, -\dfrac{5}{9}$ of the negative ones.

(2) $14, \dfrac{5}{12}$ of the positive roots; $-\dfrac{5}{161}, -104$ of the negative ones.

12. One root lies between 2 and 3, another between 1 and 2, and the 3rd between $-3$, $-4$.

13. Two real roots, one positive, the other negative.

15. $x = 2, 2, 2, -2$.

16. (1) $x = 4$. (2) $x = -8$. (3) No integral root.

17. $x = -1, -1, -1, 3$.

18. (1) one between 3 and 2, another between 1 and 0, and the 3rd between $-3$, and $-4$.

(2) one between 2 and 3, another between 3 and 4; there are no other real roots.

## TRIGONOMETRY.

2. $47\frac{3}{4}$ nearly. 3. Diameter $=.745$ of an inch nearly.

8. $A = n\pi$, or $(6n \pm 1)\dfrac{\pi}{6}$.

9. $A = 2n\pi \pm \dfrac{2}{3}\pi$, or $2n\pi$; all these values are given by the formula $A = (2n+1)\dfrac{\pi}{3} \pm \dfrac{\pi}{3}$.

10. (1) $\theta = (2n+1)\dfrac{\pi}{6}$, or $(3n \pm 1)\dfrac{\pi}{3}$.

(2) $\theta = n\pi$, or $(2n+1)\pi$, or $2n\pi \pm \cos^{-1}\dfrac{1 \pm \sqrt{(-1)}}{2}$.

(3) $\theta = n\pi + \dfrac{3\pi}{4}$, or $\theta = \frac{1}{2}\sin^{-1}2(\sqrt{2}-1)$,

where those values of $\theta$ must be chosen which satisfy the condition, $\cos\theta - \sin\theta = \sin\theta\cos\theta$.

(4) $\theta = (2n+1)\dfrac{\pi}{8}$, or $(4n-1)\dfrac{\pi}{2}$, or $\{6n + (-1)^n\}\dfrac{\pi}{6}$.

(5) $\theta = \dfrac{n\pi}{2}$, or $(4n-1)\dfrac{\pi}{16}$.

(6) $\theta = \dfrac{n\pi}{4}$, or $\dfrac{2n\pi}{3}$, or $(2n+1)\dfrac{\pi}{5}$.

(7) $\theta = (2n+1)\dfrac{\pi}{2}$, or $\{6n + (-1)^n\}\dfrac{\pi}{6}$, or $(3n \pm 1)\dfrac{2\pi}{3}$.

(8) $\theta = \dfrac{2m+1}{2(n+1)}\pi$.

11. $x = \frac{1}{2}\sin^{-1}\dfrac{4}{(2n+1)\pi}$, where $n$ is any integer except $-1$.

12. (1) $\theta = \dfrac{n\pi}{3}$, $\tan\theta = \pm\sqrt{3}$, or $\pm\sqrt{(-1)}$.

(2) $\theta = \{6m + (-1)^m\}\dfrac{\pi}{6n}$.

13. $\tan x = \pm\sqrt{\left(1 \pm \frac{2}{3}\sqrt{3}\right)}$.

14. $x = \sqrt{3}$.

15. $x = (4n+1)\frac{\pi}{2} - \theta$, and $y$ is given by $\sin(y+\theta) = \cot\theta . \cos\theta$.

17. $ab = c^2 + d^2 - a^2 - b^2$.

30. If $a$ be the distance between the two stations, the height is $\frac{a}{2} . \sin\alpha . \sec\beta$.

33. 60 ft., 100 ft., $42\frac{1}{2}$ ft.

35. $60^\circ$.

45. $A = 116^\circ, 33', 54''$. $B = 26^\circ, 33', 54''$.

63. $A = 2\sin^{-1}\frac{1}{3}$.

68. If $r$ be the radius, and $\theta$ the given angle,

$$\text{Area} = \tfrac{1}{2}\,\frac{3\sqrt{3}r^2}{\sqrt{3}\sin\theta + 2\cos^2\frac{\theta}{2}}.$$

70. $$OA^2 = \frac{r(r+p)c^2 + p(p+r)b^2 - pra^2}{(p+q+r)^2}.$$

$$OB^2 = \frac{p(p+q)a^2 + q(p+q)c^2 - pqb^2}{(p+q+r)^2}.$$

$$OC^2 = \frac{r(r+q)a^2 + q(q+r)b^2 - qrc^2}{(p+q+r)^2}.$$

71. If $a$ be the distance between the two first stations, the heights of trees are $a\sin 2\theta$, $a . \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} . \sin 2\theta$.

82. $\frac{1}{2} . \tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}$.

83. $45^\circ, 17', 6''$ nearly.

88. $\frac{\sin 2\theta}{2} - \frac{\sin 2^{n+1}\theta}{2^{n+1}}$.

90. $\tan 2^n\theta - \tan\theta$.

# ANALYTICAL GEOMETRY OF TWO DIMENSIONS.

1. (1) The straight lines $x=0$, $x-2a=0$, $x+2a=0$.
   (2) The straight lines $x-3a=0$, $x+a=0$.
   (3) The origin, and the straight lines $x-y=0$, $x+y=0$.
   (4) The straight lines $x-2a=0$, $y+3a=0$.
   (5) The straight lines $x-y=0$, $x+y-3a=0$.
   (6) The straight lines $5x-3y=0$, $5x+3y-2=0$.

2. $x=a$, $y=0$; $x=0$, $y=b$.

3. $x=3$, $y=1$; $x=2$, $y=3$; $x=1$, $y=2$.

5. $\dfrac{ab\{\sqrt{(a^2+b^2)}-c\}}{(a^2+b^2)^{\frac{3}{2}}}$.  6. $y=0$, $y+x-a=0$.

7. $99x-27y-79=0$, $21x+77y-1=0$.

8. $3x+4y-5a=0$.  9. $4(x+y)-5a=0$.

10. $b=0$, $a^2-b^2+1=0$.

11. (1) a line inclined at an angle $\alpha$ to the initial line, and terminated at the pole.

(2) a line inclined at an angle $\alpha$ to the initial line, and produced through the pole.

(3) a circle whose radius is $a$, and whose centre is in the pole, and the two lines $r+a\cos\theta=0$, $r-\dfrac{a}{\cos\theta}=0$.

12. $9x^2-y^2=4a^2$.  15. $\dfrac{\alpha}{a}-\dfrac{\beta}{b}=0$, $\dfrac{\alpha}{a}+\dfrac{\beta}{b}=1$.

19. $\dfrac{y}{x}=\dfrac{\pm\sqrt{7}-4.}{3}$.

20. $x^2+y^2-2y\beta+\dfrac{\beta^2}{2}=0$, where $\beta=2k\pm\sqrt{\{2(k^2-h^2)\}}$.

21. $x'=4$, $y'=2$.  24. $x-y=0$.

26. A Parabola.  27. $ay+bx-2xy=c\sqrt{(x^2+y^2)}$.

30. $e=\dfrac{\sqrt{3}}{2}$, $p=\sqrt{3}$.

31. (1) $\left(x-\frac{a}{2}\right)^2+4\left(y-\frac{a}{3}\right)^2=\left(\frac{5a}{6}\right)^2$, the equation to an ellipse, whose centre bisects the line joining the centre of the given ellipse with the given point, and whose axes are $\frac{5a}{3}$, $\frac{5a}{6}$, and eccentricity the same as the given ellipse, viz. $\frac{\sqrt{3}}{2}$.

(2) $(x^2+4y^2)^2=a^2(x^2+16y^2)$.

32. $y-x=0$, $9y-5x=0$, $x'=\pm 2\sqrt{\frac{7}{3}}$, $y'=2(1\mp\sqrt{\frac{7}{3}})$.

52. $\frac{x^2}{2a^2}+\frac{(y-b)^2}{2b^2}=1$, the equation to an ellipse, whose centre is at the extremity of the minor-axis.

60. $xy=c^2$, $xy=-d^2$ being the equations to the hyperbolas, and $l$, $m$ the direction-cosines of the given line, the equation to the locus is

$$2(ly+mx)^2(2xy-c^2+d^2)+lm(c^2+d^2)^2=0.$$

65. (1) A parabola, whose latus rectum is $c$; coordinates of vertex $0$, $-c$.

(2) Two straight lines $2x-3y+6=0$, $x+2y-2=0$.

(3) Two straight lines $2x-y+a=0$, $x-2y-a=0$.

(4) Two straight lines $\frac{x}{a}+\frac{y}{2b}+1=0$, $\frac{x}{a}-\frac{y}{2b}+1=0$.

(5) An ellipse whose semi-axes are $\frac{2\sqrt{2}a}{3}$, and $\frac{2a}{3}$; coordinates of centre $-\frac{a}{3}$, $\frac{a}{3}$.

(6) An hyperbola, whose semi-axes are $\frac{2\sqrt{2}a}{5}$, $\frac{2\sqrt{2}a}{\sqrt{5}}$; coordinates of centre $\frac{8a}{5}$, $-\frac{2a}{5}$.

(7) An equilateral hyperbola, each of whose semi-axes is $a\sqrt{2}$; coordinates of centre $0$, $-a$.

(8) An equilateral hyperbola, each of whose semi-axes is $2$; coordinates of centre $-2$, $1$.

(9) A parabola, whose latus rectum is $\frac{a}{\sqrt{2}}$; co-

ordinates of vertex, $\frac{5a}{8}$, $-\frac{a}{8}$; the parabola touches the axis of $x$ at a point $x=a$, $y=0$.

(10) An hyperbola, whose semi-axes are $\frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{8}$; $\frac{1}{\sqrt{2}}.\sec\frac{\pi}{8}$; coordinates of centre $a$, $-a$.

66. (1) $x=0$, $x=2y-4$. (2) $y=a$, $y=x-a$.

67. (1) $e=\frac{\sqrt{5}}{2}$; $x=0$, $y=\frac{3x}{4}$.

(2) $e=\frac{\sqrt{5}}{2}$; $y=2x$, $y=-2x-2$.

68. (1) $2\sqrt{(ab)}.\sin\frac{\alpha}{2}$, $2\sqrt{(ab)}.\cos\frac{\alpha}{2}$.

(2) $2a\sin\frac{\alpha}{2}$, $2a\cos\frac{\alpha}{2}$.

69. $y=x(\sqrt{3}\pm 2)$. 70. $y-x=0$.

71. $\frac{8a}{5}$; $\frac{2a}{5}$, $-\frac{3a}{5}$. 72. $\frac{8a}{13}$; $5y-12x+7a=0$.

75. (1) is the equation of two straight lines through the origin parallel to the asymptotes real, imaginary, or coincident.

(2) is the equation of the straight line joining the points at a finite distance in which the lines in (1) meet the curve.

(3) If lines be drawn parallel to $OX$ from the points of intersection of the curve with $OY$, they meet the curve in two points, the line joining which is represented by (3).

(4) Similarly for $OY$ and $OX$.

77. $\frac{n^2+3n-6}{2}$.

84. This equation corresponds to the centres of two coincident straight lines passing through $A$, $B$.

85.
$$(by-1)(b'-b)+x\{c+a(b'-b)\}=0.$$
$$(b'y-1)(b'-b)-x\{c-a(b'-b)\}=0.$$
$$ax+\tfrac{1}{2}(b+b')y=1.$$

# DIFFERENTIAL AND INTEGRAL CALCULUS.

1. (1) $\dfrac{du}{dx}=\dfrac{3x^5}{1+x^3}$. (2) $\dfrac{du}{dx}=\dfrac{2a}{x^2-a^2}$.

(3) $\dfrac{du}{dx}=\dfrac{2x}{\sqrt{(x^2+a^2)}\sqrt{(x^2+b^2)}}$. (4) $\dfrac{du}{dx}=\dfrac{225x^{-6}}{\sqrt{(1+x^2)}}$.

(5) $\dfrac{du}{dx}=x^{\sin x}\left\{\dfrac{\sin x}{x}+\log x\cos x\right\}$.

(6) $\dfrac{du}{dx}=e^{\frac{\tan x}{x}}\left\{\dfrac{x\sec^2 x-\tan x}{x^2}\right\}$.

(7) $\dfrac{du}{dx}=(\sin x)^{\log x}\left\{\dfrac{\log\sin x}{x}+\cot x.\log x\right\}$.

(8) $\dfrac{du}{dx}=(\cos x)^{\sin x}\{\cos x\log\cos x-\sin x.\tan x\}$.

(9) $\dfrac{du}{dx}=(\tan x)^{\cot^{-1}x}\left\{2\operatorname{cosec}2x\cot^{-1}x-\dfrac{\log\tan x}{1+x^2}\right\}$.

(10) $\dfrac{du}{dx}=\dfrac{1+x^2}{1-x^2+x^4}$. (11) $\dfrac{du}{dx}=-\operatorname{cosec}^2 x$.

(12) $\dfrac{du}{dx}=2\sqrt{(a^2-x^2)}$. (13) $\dfrac{du}{dx}=-\dfrac{x^2e^x}{(1+x)^{\frac{3}{2}}\sqrt{(1-x)}}$.

(14) $\dfrac{du}{dx}=-\sin x.\sin(\cos x).\sin\cos(\cos x)$.

(15) $\dfrac{du}{dx}=\dfrac{\sin 2x}{\sqrt{(\cos 2\alpha-\sin 2x)}}$.

2. (1) $\dfrac{du}{du}=\dfrac{\cos x}{1+\sin x}$, $\dfrac{du}{dx}=\dfrac{3}{1+x^2}$.

3. $x=\dfrac{n\pi}{2}$, or $n\pi+(-1)^n\dfrac{\pi}{6}$. 4. $2\sqrt{x}.e^x$.

9. (1) $2(-1)^{r-1}\dfrac{\lfloor\!\underline{r-2}}{x^{r-2}}$. (2) $2^r\left(\dfrac{1}{3}\right)^{\frac{r}{2}}e^{\frac{x}{\sqrt{3}}}\cos\left(x+r\dfrac{\pi}{3}\right)$.

(3) $e^x\{x^3+3rx^2+3r(r-1)x+r(r-1)(r-2)\}$.

(4) $(-1)^r.\underline{|r}\left\{\frac{1}{(x-2)^{r+1}}-\frac{1}{(x-1)^{r+1}}\right\}$.

11. This is a ramphoid cusp touching the axis of $x$ at a point $x=a$.

12. $1+x-x^2-\frac{4x^3}{3}-\&c. \ldots$

$$2^n.\cos n\frac{\pi}{3}.\frac{x^n}{\underline{|n}}.$$

13. $-\left\{\frac{x^2}{\underline{|2}}+\frac{2x^4}{\underline{|4}}+\frac{16x^6}{\underline{|6}}+\frac{272x^8}{\underline{|8}}+\ldots\right\}$.

14. $2^n\left\{1+\frac{nx^2}{1.2}+n(3n-2)\frac{x^4}{1.2.3.4}+\ldots\right\}$.

16. $\log\tan\frac{m}{2}+e+\cos m.\frac{e^2}{1.2}+2\cos 2m\frac{e^3}{1.2.3}$
$$+\frac{3}{4}(9\cos 3m-\cos m)\frac{e^4}{1.2.3.4}+\ldots.$$

18. $\sin\theta+e^\theta.\cos\theta\, x+e^{2\theta}(2\cos\theta-\sin\theta)\frac{x^2}{1.2}+\ldots.$

19. (1) 1. (2) 2. (3) 2. (4) $\frac{2}{3}$.
(5) $-2(1+2\pi)$. (6) $-\frac{4}{9}$. (7) 2. (8) 1.
(9) $-\frac{4}{\pi^2}$. (10) $\frac{2}{5}$. (11) 1. (12) $-\frac{1}{2}$.
(13) $-\frac{\pi^2}{8}$. (14) $\frac{1}{6}$. (15) $\infty$. (16) 0.

20. $-\frac{b}{a^2}.\frac{1}{\sin^3 t}$. 21. $\frac{1}{x}\left(\frac{d^2y}{dt^2}+y\right)$.

26. $-\dfrac{\frac{d^2x}{dy\,dz}}{\left(\frac{dx}{dz}\right)^3}=0$.

27. (1) $x=1$ gives a minimum, $x=-\frac{1}{3}$ a maximum.

(2) $x=2a$ gives a minimum, $x=\frac{8a}{5}$ a maximum.

(3) $x = -\frac{\sqrt{6}+1}{5}$ gives a min., $x = \frac{\sqrt{6}-1}{5}$ a max.

(4) $x = a$ gives a minimum, $x = \frac{3a}{5}$ a maximum.

(5) $x = a$ gives a minimum, $x = -\frac{a}{7}$ a maximum.

(6) $x = n\pi$ gives a min., $x = 2n\pi \pm \cos^{-1}\frac{1}{4}$ a maximum.

(7) $x = \frac{2n\pi - a}{2}$ gives a min., $x = \frac{(2n+1)\pi - a}{2}$ a max.

(8) $\left.\begin{array}{l} x = (2n+1)\pi \\ \text{or } x = 2n\pi \pm \frac{\pi}{4} \end{array}\right\}$ gives a minimum,

$\left.\begin{array}{l} x = 2n\pi \\ \text{or } x = (2n+1)\pi \pm \frac{\pi}{4} \end{array}\right\}$ a maximum.

(9) $x = \pm \frac{1}{\sqrt{2}}$ gives a maximum.

32. The hypothenuse of the triangle is the minimum axis, in which case the ellipse becomes a circle. The maximum value is half the hypothenuse and is the axis of an equilateral hyperbola.

38. (1) $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{d^3y}{dx^3} - \frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0.$

(2) $\cos y \frac{dz}{dx} - \sin y . \frac{dz}{dy} = 0.$

(3) $\frac{dz}{dx} - y . \log y . \frac{dz}{dy} = 0.$ (4) $\frac{dz}{dx} - \frac{dz}{dy} = 0.$

(5) $x\frac{dz}{dx} + y\frac{dz}{dy} = z.$

(6) $\frac{1}{x^2} . \frac{d^2z}{dx^2} - \frac{1}{y^2} . \frac{d^2z}{dy^2} = \frac{1}{x^3} . \frac{dz}{dx} - \frac{1}{y^3} . \frac{dz}{dy}.$

(7) $xy\frac{d^2z}{dx\,dy} - x\frac{dz}{dx} - y\frac{dz}{dy} + z = 0.$

(8) $\frac{d^2z}{dx\,dy} . \frac{dz}{dx} - \frac{dz}{dy} . \frac{d^2z}{dx^2} = 0.$

40. (1) $x=a$, $y=-x-a$, and three coincident asymptotes given by $y=0$.

(2) $y+x=0$, $y-x+1=0$, $x=0$; the curve touches the last on both sides in the positive and negative direction at infinity.

(3) $x=4a$, $y=x-4a$, $y=2(x+8a)$.

(4) $x=0$, $y-2x=0$, $y+x=0$.

(4) has two branches lying entirely between $x=0$, and $y-2x=0$, and has two conjugate points at infinity where $y+x=0$ meets it.

45. The origin is a point of inflexion.

46. The origin is a double point.

47. There is a cusp of the first species at the point $y=a$, $x=-\frac{3a}{2}$.

48. $$A=-2\,\frac{3m^2-2m+1}{m^2}\,a.$$

$$B=\;2\,\frac{m^2-m+1}{m(m-1)}\,a.$$

$$C=\;2\,\frac{m^3-3m^2+2m-1}{m^2(m-1)}.$$

49. See an Article "on certain points of singular curvature in plane curves," in the Quarterly Journal of Pure and Applied Mathematics, May 1857, by E. Walker, Esq., Trinity college.

52. $p_1=r_1\sin\alpha$.

54. $r\theta e+c=0$.

55. If the centre of the given circle be the origin and the fixed point in the initial line, the equation is

$$\theta-\cos^{-1}\frac{a}{r}=-\frac{\sqrt{(r^2-a^2)}}{a}.$$

58. The curve (9) cuts the asymptote at a point

$$x=-\frac{2a}{3},\quad y=\frac{4a}{3}.$$

60. (1) $\log\dfrac{x+1}{\sqrt[4]{(x^4+2x^3)}}-\dfrac{1}{2x}$.

(2) $2\log\dfrac{x}{x-1}-\dfrac{1}{x}-\dfrac{1}{x-1}$.

(3) $\log\sqrt{(x+2)}\,.\sqrt[4]{(x^2+4)}+\dfrac{1}{2}.\tan^{-1}\dfrac{x}{2}$.

(4) $\dfrac{1}{50}\log\dfrac{x^2+1}{(2-x)^2}+\dfrac{7}{25}\tan^{-1}x+\dfrac{2x-1}{10(x^2+1)}$.

(5) $\dfrac{1}{3}\log\left\{\dfrac{\sqrt[3]{(a^3+x^3)}}{x}-1\right\}-\dfrac{1}{6}\log\left\{\left(\dfrac{\sqrt[3]{(a^3+x^3)}}{x}+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\right\}$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2\sqrt[3]{(a^3+x^3)}}{x\sqrt{3}}+\frac{1}{\sqrt{3}}\right\}.$$

(6) $\dfrac{3x-2x^3}{3(1-x^2)^{\frac{3}{2}}}$. (7) $\dfrac{x^6-6x^4-24x^2-16}{3(1+x^2)^{\frac{3}{2}}}$.

(8) $\dfrac{1}{\sqrt{2}}\log\left\{\dfrac{x}{2+x+\sqrt{(4+4x+2x^2)}}\right\}$.

(9) $\dfrac{1}{\sqrt{(-3)}}\log\left\{\dfrac{1+x}{\sqrt{(9-12x-12x^2)}-5-2x}\right\}$.

(10) $\dfrac{2}{3}\tan^{-1}\dfrac{\tan x}{2}-\dfrac{x}{3}$.

61. $\dfrac{16a^2}{3}$. 62. $\dfrac{a^2}{\sqrt{3}}\left(\dfrac{\pi}{\sqrt{3}}+1\right)$.

63. (1) $\dfrac{\pi a^2}{3}$. (2) $\pi a^2$.

66. If $a$ be the side about which the parallelogram revolves, $b$ the adjacent side, and $A$ the angle between them,

$$\text{Volume}=\pi ab^2\sin A,\quad \text{Surface}=2\pi b(a+b\sin A).$$

$$\text{Area}=\frac{c^2}{4\alpha}\log\frac{\pi+2\alpha}{\pi+\alpha}.$$

67. $\dfrac{3\pi a^4}{4}$.

## DIFFERENTIAL EQUATIONS.

1. $(ax+by)xy - x^m = C.$ 2. $\frac{1}{y^2}.$

3. (1) $(x+y)^{n-m} = C.(x-y)^{n+m}.$
(2) $y = Ce^{\sin^{-1}x} - \sin^{-1}x - 1.$
(3) $a\log(x-y-a) = y + C.$
(4) $x^3y = -x^2\cos x + 2x\sin x + 2\cos x + C.$
(5) $y = \cos x\left\{\frac{x^{m+1}}{m+1} + C\right\}.$
(6) $e^{\frac{y}{x}} = \log\frac{x}{C}.$

4. (1) $x^2 + y^2(\log y - \frac{1}{2}) = C.$ (2) $x^2 + y^2 = cy.$

8. $x^2 = 4c(c+y).$

9. (1) $\log y.\cos(x-c) = c_1.$
(2) $y = c + c_1e^{x(1+\sqrt{2})} + c_2e^{x(1-\sqrt{2})} - 2x.$
(3) $y = \frac{x^m}{(m-1)^2} + e^x(c + c_1x).$
(4) $y = c\cos 2x + c_1\sin 2x + c_2\cos 3x + c_3\sin 3x + \frac{1}{20}x\sin 2x.$
(5) $y = -\frac{e^{-mx}}{(m-2)^2(m+2)} + ce^{2x} + e^{-2x}(c_1 + c_2x).$

10. (1) $xy - az = c.$
(2) $e^{\log x}\log y + e^{\log y}\log z + e^{\log z}\log x = c.$
(3) $z = \tan^{-1}x + \frac{x^2}{y^2} + c.$

The factor required is $e^{x^2}$.

11. (1) $x + y = f\left(\frac{x}{z}\right).$ (2) $ye^{-nz} = f(xe^{-mz}).$

(3) $\dfrac{x+y+z}{(x-y)^2}=f\{x(x-2z)-y(y-2z)\}.$

(4) $\dfrac{x^2+y^2+z^2}{(x^2-y^2)^2}=f\{x^2(x^2-2z^2)-y^2(y^2-2z^2)\}.$

(5) $$z\left\{\frac{x^{l+1}}{l+1}+\frac{y^{m+1}}{m+1}+\frac{z^{n+1}}{(n+1)(n+2)}\right\}-\frac{x^{l+1}}{l+1}=f\left\{\frac{x^{l+1}}{l+1}+\frac{y^{m+1}}{m+1}\right\}.$$

(6) $x=f(y)+\phi(z).$

(7) $z=f(x-y)+\phi(bx-ay).$

(8) $z=f(x+z)+\phi(y+z).$

12. See Tait's *Dynamics*, page 14.

13. $u_x=(c+c_1x)+c_2(-5)^x.$

15. $\dfrac{1}{2\sqrt{6}}\left\{\dfrac{(2+\sqrt{6})^{x+1}-1}{1+\sqrt{6}}-\dfrac{(2-\sqrt{6})^{x+1}-1}{1-\sqrt{6}}\right\}.$

---

## GEOMETRY OF THREE DIMENSIONS.

1. $x=a$ represents a plane parallel to the plane of $yz$.

$y=nz+b$ represents a plane parallel to the axis of $x$, whose trace on the plane of $yz$ is the line $y=nz+b$.

$y=mx$ represents a plane through the axis of $y$.

(2) and (3) taken together represent the straight line in which they intersect.

(1), (2), (3) taken together represent the point whose coordinates are

$$x=a,\quad y=nma+b,\quad z=ma.$$

2. If $a$, $b$, $c$ be the coordinates of the given point $l$, $m$, $n$; $l'$, $m'$, $n'$ the direction-cosines of the given lines, the required equation is

$$\frac{x-a}{mn'-m'n}=\frac{y-b}{nl'-n'l}=\frac{z-c}{lm'-l'm}.$$

3. $(c^2+r^2-a^2)z+2acx=C.$

5. If the equations to the lines be $x=mz+h$, $y=0$; $y=nz+k$, $x=0$, and those of the circle $x^2+y^2=a^2$, $z=0$, the equation to the surface is

$$x^2(hy+kx-hk-kmz)^2+y^2(hy+kx-hk-hnz)^2$$
$$=a^2\{xy-(x-h-mz)(y-k-nz)\}^2.$$

6. $x^2\left\{\frac{bc-bz-cy}{y-b}\right\}^2+c^2y^2=a^2(z-c)^2.$

This problem is deducible from (5) when in the notation of (5) we put $n=0$, $k=b$, $m=-\frac{h}{c}$, and then make $h=\infty$.

7. If the line to which the generating line is perpendicular be taken for the axis of $z$, and the equations to the other line be $x=mz+a$, $y=nz+b$, the equation required is

$$z(nx-my)=ay-bx.$$

8. $\cos\alpha$. 9. $x^2.\tan^2\alpha+y^2\tan^2\beta=z^2$.

11. A concentric ellipsoid whose semi-axes are $\frac{a^2}{r}$, $\frac{b^2}{r}$, $\frac{c^2}{r}$, where $r$ is the radius of the sphere, and $a$, $b$, $c$ the semi-axes of the first ellipsoid.

18. The axis of $z$; a plane.

20. $8\pi a^3$. 21. $\frac{a^3}{4}\left(\pi+\frac{3\sqrt{3}}{4}\right)$. 22. $\pi c^2\sqrt{2}$.

23. Two parallel planes whose equations are $x+2y-3z+4=0$, $x+2y-3z-4=0$.

24. If $a$ be the distance of the point from the vertex of the cone, and $\alpha$ the semi-vertical angle of the cone, the distance required is $2a\sin(\pi\sin\alpha)$.

26. If $Ax+By+Cz+D=0$, be the equation to the given plane, $(h, k, l)$ the coordinates of the given point, $\alpha$ the given angle, the required equation is

$$\frac{\{A(x-h)+B(y-k)+C(z-l)\}^2}{(x-h)^2+(y-k)^2+(z-l)^2}=(A^2+B^2+C^2)\sin^2\alpha.$$

## ELEMENTARY STATICS.

1. If $P$ and $Q$ be the two forces, the angle is

$$\cos^{-1}\frac{3(P^2+Q^2)-8PQ}{2PQ}.$$

3. The distance of the point from one extremity is $\frac{1}{6}$th of the length of the rod.

4. $\frac{3W}{2}$, where $W$ is the weight of the shorter arm.

6. One side of the square is inclined at an angle $\cot^{-1}2$ to the horizon.

7. $60°$.

8. $\frac{W}{2\sqrt{3}}$, where $W$ is the weight of the upper cylinder.

9. $\frac{4W-7P}{16}$, $\frac{4W-35P}{16}$, where $P$ is the power, $W$ the weight.

16. $\tan^{-1}\left\{\mu.\frac{W+P}{P}\right\}$, where $W$ is the weight of the wedge, $P$ that of the sphere.

17. At an angle $\tan^{-1}\frac{1}{3\sqrt{3}}$ to the horizon.

19. If $\alpha$ be the angle which the line joining the centres of the spheres makes with the horizon, the force $= W\cot\alpha$, where $W$ is the weight of one of the spheres. In the latter case the line joining the centres makes an angle $\cot^{-1}6\mu$ with the horizon.

20. If the weight be moved a distance $c$, and $a$ be the side of the triangle, $W$ one of the weights, the force will be

$$\left\{\frac{a+2c\sqrt{3}}{\sqrt{(a^2+3c^2+ac\sqrt{3})}}-1\right\}W.$$

23. If $a$ be one of the sides of the triangle, the centre of gravity is at a distance $\frac{11-3\sqrt{3}}{7\sqrt{3}}a$, from the base.

30. A circle, the distance of whose centre from the vertex is $\frac{2}{3}$ of the radius of the given circle.

31. $2P$. 32. $\dfrac{\mu a e^{\frac{\mu\pi}{2}}}{1+\mu e^{\frac{\mu\pi}{2}}}$, $a$ being the length of the string. [This Problem should have been placed in *Analytical Statics*.]

37. The equilibrium is unstable.

43. If $\theta$ and $\phi$ be the angles which the two portions of the string make with the vertical, and $m$ the ratio of the weight of the pully to that of the weight,

$$\cos\theta = \frac{m^2-8}{2m}, \quad \cos\phi = \frac{m^2+8}{6m}; \quad m > 2\sqrt{2} < 4.$$

44. If $4a$ be the latus rectum of the parabola, $2b$ the length of the rod, $c$ the distance between the hinge and vertex, and $W$ the weight of the rod, the force $= \dfrac{ba^{\frac{1}{2}}}{2(a+c)^{\frac{3}{2}}}W$.

50. The angle which each rod makes with the horizon is $\tan^{-1}\dfrac{W+2P}{2P}$, where $W$ is the weight of one of the rods, and $P$ that of either weight. The position of equilibrium is unstable.

51. $\dfrac{3a(1-\mu).W}{c-2a(1-\mu)}$, $W$ being the weight of one of the rods, $2a$ its length, $c$ the radius of the sphere, and $\mu$ the coefficient of friction. The Problem is impossible if the sphere be smooth.

56. The highest and lowest positions are stable, the third unstable.

## ANALYTICAL STATICS.

6. The positions of equilibrium are given by $x=y=\frac{c}{n\sqrt{2}}$, except when $n=2$, for in that case the equation $x^{n-2}=y^{n-2}$ becomes indeterminate. The curve then becomes a circle, and we know that a body will rest at any point of a circle $x^2+y^2=c^2$ if acted on by forces parallel to the axes and proportional to the coordinates of that point.

11. An angle $\tan^{-1}\frac{2}{\sqrt{3}}$, with the vertical. 12. 2 : 1.

13. The whole pressure will be at $A$, and equal to half the weight of the square.

15. (1) If $2\alpha$ be the angle of the sector, the distance from the centre is $\frac{n+2}{n+3}\cdot\frac{\sin\alpha}{\alpha}\cdot a$, $a$ being the radius.

(2) If $A$ be the given point, $C$ the centre, $AC=a$, and $CG$ be taken $=\frac{a}{\pi}$ and in a direction perpendicular to $AC$, $G$ is centre of gravity.

(3) The distance of the centre of gravity of the whole arc from the pole is $\frac{4a}{5}$.

17. $\bar{x}=\frac{25}{56}h$, $\bar{y}=\frac{25}{56}k$, $h$, $k$ being the coordinates of the point of intersection of the curves.

18. Its distance from the pole is $\frac{512a}{315\pi}$.

19. If $a$ be the radius of the circle, $2\alpha$ the vertical angle, $\bar{z}=\frac{3}{32}\pi a\cot\alpha$.

20. (1) $\frac{\pi d^3}{2}\cot^2\frac{A}{2}$, $d$ being the given diagonal, $A$ the given angle.

(2) $\frac{2176}{15}\pi a^3$, $4a$ being the latus rectum.

21. $\frac{\pi a^3}{3\sqrt{2}}$, $a$ being a side of the square.

22. Force : weight of string :: 2.003 : 1 nearly.

29. Half the weight of the chain.

37. If $2\theta$, $2\phi$, be the angles which the upper and lower void arcs subtend at the centre, and $W$, $P$ the weights of the sphere and ring,

$$\tan\theta - \theta = \frac{\pi}{\sin\theta}\left\{1+\frac{P}{W}\right\},\quad \tan\phi - \phi = \frac{\pi}{\sin\phi}\cdot\frac{P}{W}\bigg\}.$$

31. $\frac{1}{2}\left(\frac{\log n}{\mu\pi}-1\right)$, both strings being vertical.

32. If $P_1$, $P_2$ be the greatest and least values of $P$, $\frac{P_1}{P_2}=e^{\mu(\pi+3\alpha)}$.

33. 60°.

40. If $P$, $Q$ be the weights of the two rods; $a$, $b$ their lengths, $c$ the distance between the hinges, $r$ the radius of the hemisphere, the angle which the first makes with the horizon is $\sin^{-1}\left[\frac{r.\{\sqrt{(Pa)}+\sqrt{(Qb)}\}}{c\sqrt{(Pa)}}\right]$.

43. It must be just greater than $\frac{5a}{3}$, $a$ being the radius of the hemisphere.

47. $\theta=0$.

## ELEMENTARY DYNAMICS.

2. $m+n^2g$.

4. (1) 1610. (2) $\left(\frac{360}{161}\right)^{\frac{1}{2}}$ seconds. (3) 57.5 feet.

5. 30°. 8. 69. 12. $\frac{g}{6}$.

20. Each of the balls moves back with $\frac{5}{6}$ths of its original velocity.

25. If $x$ be the abscissa of the point, and $4a$ the latus rectum, the range is $\frac{16ax(x-a)}{(x+a)^2}$.

27. The ball rises to a point whose distance below the focus is $\frac{1}{4}$th of the latus rectum.

29. $\frac{4}{3e-1}$, $e$ cannot be greater than $\frac{1}{3}$. When $e=\frac{1}{3}$, $n$ becomes infinite, in which case the impact is the same as if $nm$ was a hard plane. The condition that a ball after striking a hard plane should go off at 60° is $e=\frac{1}{3}$.

38. In both cases the angle is $\frac{1}{2}\sin^{-1}\frac{1}{4}$.

41. If $v\frac{M}{m}$, $v\frac{M}{m'}$ be the velocities of the balls, $2\alpha$ the angle which their directions make with each other, the velocity is

$$2v\cos\alpha.\frac{M}{m+m'}.$$

48. $M$ moves back with half its original velocity; $m$ moves back with $\frac{3}{2}$ of $M$'s original velocity. The impact is direct.

50. In the latter case, let $x$ be the distance of the point of projection from the wall, $a$ the breadth of the room; then if $x>\frac{2a}{3}$, the ball hits the ceiling before the wall; if $x=\frac{2a}{3}$, the ball hits the top of the wall; if $x<\frac{2a}{3}$, the ball hits the wall before the ceiling.

51. $\theta = n\pi$, or $n\pi \pm \sqrt{\frac{13}{3}}$, or $n\pi \pm \sqrt{\frac{5}{3}}$.

52. It will rise to a point whose distance from the lowest point is $\frac{5}{8}$ths of one of the sides of the hexagon.

53. If $\theta$, $\phi$ be the inclinations of the upper and lower portions of the string to the vertical; $a$, $b$ the lengths of the upper and lower portions; $P$, $Q$ the upper and lower weights; $\omega$ the angular velocity,

$$g \tan\phi - \omega^2(a \sin\theta + b \sin\phi) = 0.$$

$$P\omega^2 a \sin\theta - (P + Q)\, g \tan\theta - Q\, g \tan\phi = 0.$$

---

## NEWTON, I., II., III.

1. $\frac{1}{2}$, 2. 3. $\frac{4}{3}$.

5. $\frac{3}{4}$ths of the circumscribing parallelogram, one of whose sides touches the curve at the vertex.

6. Its distance from the vertex is $\frac{3}{8}$ths of the side of the square.

7. $\dfrac{\pi a^2}{12b^2\sqrt{(a^2+b^2)}}\{7b^4 - 2a^2b^2 - a^4\}$, $a$ and $b$ being two of the sides.

19. $8000.\left(\dfrac{\alpha^2}{a\pi^2}\right)^{\frac{1}{3}}$ miles, $\alpha$ being the number of seconds in 27 days, 7 hours, and $a$ the number of feet in 4000 miles. This may be shewn to be about 240000 miles.

20. $1200\pi^2 : g$. 21. $30°$.

23. The tension varies inversely as the distance of one of the rings from the nearer peg.

27. An ellipse whose eccentricity is $\dfrac{\sqrt{3}}{2}$.

44. 225 days, nearly.

52. It is equal to the major axis of the ellipse.

## DYNAMICS OF A PARTICLE.

2. $\pi\sqrt{\left(\dfrac{2am}{3gM\sqrt{3}}\right)}$, $m$ being the mass of the particle, $M$ that of one of the weights.

5. The distance of the particle from $S$ at the time $t$ is $\dfrac{\mu\mu' t^2 - 2\mu(1-\cos\sqrt{\mu'}t)}{\mu'^2}$. If $\mu'=0$, or the repelling force be the only one which acts, the above expression may easily be shewn to be equal to $\dfrac{\mu t^4}{12}$.

9. Force $\propto$ inversely as the cube of the distance.

10. A parabola.

11. Force $\propto$ inversely as the cube of the distance measured along the arc from the vertex.

12. If the initial line be that which joins the centre of force and the point of projection, the equation to the orbit is. $\dfrac{a^2}{r^2} = \tan\left(2\theta + \dfrac{\pi}{4}\right)$.

14. The body must be depressed through $2a$, $a$ being the natural length; $\pi\sqrt{\dfrac{a}{g}}$.

---

## RIGID DYNAMICS.

1. (1) Mass. $\dfrac{a^2}{12}$.

(2) (1) Mass. $\left\{\dfrac{1+2\cos^2\alpha}{2} - \dfrac{3\sin 2\alpha}{4\alpha}\right\}a^2$.

(2) Mass. $\left\{\dfrac{1+2\cos^2\alpha}{2} - \dfrac{3\sin 2\alpha}{2\alpha} - \dfrac{1+14\cos^2\alpha}{4\alpha^2}\right\}a^2$.

(3) Mass. $\frac{8a^2}{35}$. (4) Mass. $\frac{3a^2}{8}$. (5) Mass. $\frac{\pi a^2}{8}$.

5. If $\omega$ = angular velocity in lowest position.

$P_1$ = pressure in highest position.

$P_2$ = pressure in lowest position.

$a$ = radius.

$$\omega^2 = \frac{16g}{5a}. \quad \frac{P_2}{P_1} = \frac{21}{5}.$$

6. When horizontal the ratio is 1 : 3, when vertical 7 : 3.

7. Supposing the plane of the disks perpendicular to the plane containing the tangent and the axis of suspension the time $= \frac{\pi}{2}\left\{\frac{5(a^4+c^4)+4(a^2+c^2).b^2}{g(a^2+c^2)(b^2+d^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$, $a$, $c$ being the radii of the disks, and $b$, $d$ being the respective distances of the axis of suspension and centre of gravity of the disks from the point of contact.

8. $\pi\sqrt{\left(\frac{2a}{3g}\cdot\frac{\pi+2}{2+\sqrt{3}}\right)}$, $a$ being the radius.

9. The distance is $\frac{1}{7}$th of the latus rectum from the focus.

10. The equation to the line is $x + 3y = 0$.

11. If the centre of the rectangle be the origin, and the axes of $x$ and $y$ be perpendicular to the sides, the equation to the axis is $\frac{3hx}{a^2} + \frac{3ky}{b^2} + 1 = 0$, $h$, $k$ being the coordinates of the point of impact and $2a$, $2b$ the sides of the rectangle. This equation shews that the axis is the polar of the point $(-3h, -3k)$ with reference to the ellipse which touches the rectangle at the middle points of the sides.

13. $\frac{1}{4}$th of the weight of the square.

18. If the axis of $y$ be parallel to the diameter about which the sphere has the angular velocity $\omega$, and $v$ be the velocity of the plane supposed in the plane of $xy$, $a$ the radius of the sphere, $\alpha$ the angle which the initial direction makes with the axis of $x$, $\tan\alpha = \frac{v}{a\omega}$. The subsequent path will be a straight line.

19. If $a$ be the radius, it will roll through an arc $a\cos^{-1}\dfrac{10}{17}$, before it leaves the sphere.

22. The time of a small oscillation is

$$2\pi.\left\{\frac{(M+3m\cos^2\alpha)a}{3(M\sin 2\alpha - m\sin\alpha)g}\right\}^{\frac{1}{2}},$$

where $M\cos 2\alpha = 2m\cos\alpha$, $2a$ being the length of the rod, $M$ its mass, and $m$ the mass of the weight.

28. Velocity at the time $t = \dfrac{2(M+m)}{3M+2m}g\sin\alpha\, t$, $M$ being the mass of the fixed cylinder, $m$ that of the other, and $\alpha$ the inclination of the plane to the horizon.

29. If $V$ be the velocity of the centre before impact,
$v$ be the velocity of the centre after impact,
$\omega$ angular velocity round the centre after impact,
$e$ the eccentricity,
$p$, $q$ the perpendiculars from the centre on the tangent and normal at the point of impact,

$$v = \gamma - \frac{(1+e)(p^2+k^2)V}{p^2+k^2+q^2}, \quad \omega = \frac{(1+e)Vq}{p^2+k^2+q^2}.$$

30. If $2b$ be the distance between $A$ and $B$, $b$ must be less than $k$.

31. If $M$ be the mass of the hollowed cube,
$m$ be the mass of the particle,
$a$ the radius of the cavity,
the velocity of projection is that due to the height

$$\frac{5M+4m}{2M}.a.$$

32. The direction makes an angle $\tan^{-1}\dfrac{3\sin^2\alpha+2}{3\sin\alpha\cos\alpha}$ with $AB$; $\dfrac{2P\sqrt{(4+21\sin^2\alpha)}}{16+9\sin^2\alpha}$, $P$ being the blow.

## HYDROSTATICS.

1. The volumes are as 57 : 1, the weights as 4446 : 97.

2. The volumes are as 5 : 84.

3. $4\dfrac{91}{216}$ ounces.

6. The inclination of the line joining the centre and upper surface to the horizon is 30°.

9. $\sin^{-1}\dfrac{2}{7}$.

10. $g\rho\dfrac{P^3}{81}$, $P$ being the perimeter, and $\rho$ the density of the fluid.

18. $\dfrac{a}{2}$, $a$ being the radius.

29. $\frac{1}{2} W \tan\alpha \dfrac{\sigma - \rho}{\sigma}$, $W$ being the weight of each rod, $2\alpha$ the angle between them, and $\sigma$, $\rho$ the densities of the fluid and solid.

30. The radius of the heaviest sphere : the radius of the given sphere 1 : $\sqrt{2}$, and its density is half that of the fluid.

31. The distance from the centre is $\dfrac{3}{16\sqrt{2}}(\pi + 2)$.

32. The equation to the cardioid being $r = 2a\cos^2\dfrac{\theta}{2}$, the required depth is $\dfrac{63\pi a}{256}$.

33. The cylinder will be just immersed.

35. After 522 strokes. The cylinder is supposed to float in a vessel of just the same section as the receiver, which fits on the top of the vessel.

38. If $\alpha$, $\beta$ be the portions of the axis and base immersed, $\rho$, $\sigma$ the densities of the cycloid and fluid, and $a$ the radius of the generating circle, the values of $\alpha$ and $\beta$ are given by the equations,

$$\frac{\beta^2 - \alpha^2}{3} = a\beta\left(\frac{\pi}{2} - \frac{8}{9\pi}\right) - \frac{5a\alpha}{6}, \quad \alpha\beta\sigma = 3\pi a^2\rho.$$

39. $HM = \frac{5\sqrt{5}}{24} AC$, $H$ being the centre of gravity of the fluid displaced.

40. If $2\alpha$ be the vertical angle of the cone, $l$ the length of its axis, and $h$ that of the immersed portion, $h > l\cos^2\alpha$.

41. If $2\alpha$ be the vertical angle of the triangle, $h$ the distance of the vertex from the base, $\rho$, $\sigma$ the densities of the triangle and fluid, the least distance between the surface of the fluid and the vertex of the triangle will be

$$h.\left\{\frac{\rho - \sigma}{2\sigma}\cos^2\alpha\right\}^{\frac{1}{2}}.$$

50. $\Pi aA\left(\frac{m}{n}\log n + \frac{m-n}{n}\right)$, $A$ being the area of the piston, $a$ the length of the cylinder, $\Pi$ the atmospheric pressure.

51. If $W$ be the work done in drawing the piston up in the $n^{\text{th}}$ stroke, and $W'$ the work done in drawing the piston up if there was a perfect vacuum in the receiver,

$$\frac{W' - W}{W'} = \frac{\log_\epsilon(1+r)}{r(1+r)^{n-1}},$$

$r$ being the ratio of the volume of the barrel to that of the receiver.

54. The surface is a paraboloid of revolution, the depth of whose vertex below the surface is $\frac{\omega^2 a^2}{4g} + \frac{h}{2}$, $h$ being the height of the cylinder, $a$ the radius of its base.

55. The equation to the generating curve of the level surface is $y^2(\omega^2 - \mu) - \mu\left(z - \frac{g}{\mu}\right)^2 = c^2$, which will be an hyperbola, two straight lines, or an ellipse, according as $\omega^2 >, =, < \mu$, if $\omega^2 = 0$ the equation represents a circle. In the three last cases $c^2$ must be negative or the locus is impossible.

56. If the centre of gravity of the triangle be the origin, and the line about which the fluid is revolving be the axis of $z$, the equation to the surface is

$$(x^2+y^2)(\omega^2-3\mu)-3\mu z^2-2gz=C.$$

57. $\frac{2a}{5v}\cdot\left(2^{\frac{5}{3}}-1\right)$, $a$ being the initial radius of the annihilated sphere, and $v$ the velocity.

---

# OPTICS.

1. 10 feet.

2. $\frac{aV}{b-a}$, where $a$ and $b$ are the distances of the opaque and bright points from the plane, and $V$ the velocity of the latter.

3. 19.

9. $\frac{a}{\sqrt{2}}$, where $a$ is the radius.

10. $C\pi\sqrt{2}$.

27. $\frac{a}{2}$, where $a$ is the distance of the eye from the edge of the cup. $\frac{r}{2\sqrt{3}}$, where $r$ is the radius.

30. It divides the third chord in the ratio of 2 : 3.

33. $\frac{1}{30}$ inch; $2\frac{55}{64}$ inches from the reflector.

34. Six inches from the lens.

40. If $r$, $s$ be the radii of each lens, $a$ the distance between them, $\mu$ the index of refraction from air into the lenses, $\mu_1$ that from the lenses into the fluid, the distance of the geometrical focus from the second lens will be given by

$$\frac{1}{v}=\frac{\mu-\mu_1}{r}-\frac{\mu-1}{s}+\frac{\mu_1\{(\mu_1-\mu)r+(\mu-1)s\}}{(\mu_1-\mu)ar+(\mu-1)as+\mu_1 sr}.$$

43. $\frac{143}{12}f$.

45. Magnifying power is diminished in the ratio of 13 : 27. Field of view is increased in the ratio of 55 : 26.

46. The focal length of each is 2 inches.

47. 12 inches, $-20$ inches.

71. The logarithmic spiral.

---

## SPHERICAL TRIGONOMETRY AND ASTRONOMY.

3. $c^2 = a^2 + b^2$.

6. $BE^2 + CE^2 = 2(BD^2 + DE^2)$.

12. $\frac{1}{a} = \frac{c \cos B - b \cos C}{c^2 - b^2}$.

15. $3', 39''.8$ nearly.

16. $240^\circ$; 20 years, 60 years.

17. 8250599 miles, nearly; supposing the earth's radius equal to 4000 miles.

18. After sunset.

19. A little above the horizon in the S.E.; she will be gibbous, having moved through $\frac{3}{5}$ths of her third quarter, and the line joining the middle points of the circular and elliptic arcs will be inclined at about $45^\circ$ to the horizon.

21. $47^\circ\ 51'\ 19''5$ nearly.

23. The star will be visible shortly after the Sun sets in the N.E., and will remain visible till the Sun rises, when it will appear in the S.S.W., so that it will be seen from about 20 minutes past eight in the evening till five in the morning.

25. $12\frac{3}{5}$ feet nearly; twice, when $\sin \odot = \tan 15 \cot \omega$.

26. Change in azimuth = change of declination

$$\times \cot \omega \sec \odot \operatorname{cosec} l.$$

32. If $\alpha, \alpha'$; $\beta, \beta'$ be the altitudes of the stars at the two stations, the inclination of the meteor's path to the horizon is

$$\tan^{-1}\left\{\frac{\sin\alpha.\sin\beta\sin(\alpha'-\beta')-\sin\alpha'.\sin\beta'.\sin(\alpha-\beta)}{\sin\alpha'.\cos\beta'\sin(\alpha-\beta)-\sin\alpha.\cos\beta.\sin(\alpha'-\beta')}\right\}.$$

At the first observation the height is $\dfrac{a\sin\alpha.\sin\beta}{\sin(\alpha-\beta)}$, $a$ being the distance between the stations.

JONATHAN PALMER, PRINTER, CAMBRIDGE.

CAMBRIDGE SENATE-HOUSE PROBLEMS:

1848—51 (PROBLEMS). With Solutions by FERRERS and JACKSON. 15*s*. 6*d*.

1848—51 (RIDERS). With Solutions by JAMESON. 7*s*. 6*d*.

1854 (PROBLEMS AND RIDERS). With Solutions by WALTON and MACKENZIE. 10*s*. 6*d*.

1857 (PROBLEMS AND RIDERS). With Solutions by CAMPION and WALTON. 8*s*. 6*d*.

---

MR. DRAKE'S EUMENIDES OF ÆSCHYLUS. With English Notes. 7*s*. 6*d*.

MR. DRAKE'S DEMOSTHENES DE CORONA. With English Notes. 5*s*.

MR. FROST'S THUCYDIDES, BOOK VI. With English Notes. 7*s*. 6*d*.

DR. HUMPHREY'S EXERCITATIONES IAMBICÆ. Or, Exercises in Greek Iambic Verse, with Rules. SECOND EDITION. 5*s*. 6*d*.

MR. MAYOR'S JUVENAL FOR SCHOOLS. 10*s*. 6*d*.

MR. MERIVALE'S (Historian of Rome) SALLUST FOR SCHOOLS. SECOND EDITION. 5*s*.

THE CATILINE and the JUGURTHA may also be had separately, price 3*s*. each.

MR. THRING'S CONSTRUING BOOK. 2*s*. 6*d*.

MR. WRIGHT'S (Head Master of Sutton Coldfield School) HELLENICA: A FIRST GREEK CONSTRUING BOOK. With English Notes and a Vocabulary. SECOND EDITION. 3*s*. 6*d*.

MR. WRIGHT'S HELP TO LATIN GRAMMAR. 4*s*. 6*d*.

MR. WRIGHT'S SEVEN KINGS OF ROME: A FIRST LATIN CONSTRUING BOOK. SECOND EDITION. With English Notes. 3*s*.

MR. WRIGHT'S VOCABULARY and EXERCISES for the above. 2*s*. 6*d*.

---

MR. PARMINTER'S MATERIALS FOR ENGLISH GRAMMAR. 3*s*. 6*d*.

MR. THRING'S (Head Master of Uppingham School) ELEMENTS OF ENGLISH GRAMMAR. NEW EDITION. 2*s*.

MR. THRING'S CHILD'S ENGLISH GRAMMAR. New EDITION. 1*s*.

---

MR. CROSSE'S ANALYSIS OF PALEY'S EVIDENCES. 3*s*. 6*d*.

MR. HARDWICK'S HISTORY OF THE CHURCH, from A.D. 590 to 1600. 2 Vols. 21*s*.

Each Volume may be had separately, price 10*s*. 6*d*.

MR. PROCTER'S MANUAL OF THE PRAYER-BOOK. THIRD EDITION. 10*s*. 6*d*.

MR. RAMSAY'S MANUAL OF THE CHURCH CATECHISM. 3*s*. 6*d*.

MR. SIMPSON'S CHURCH HISTORY OF THE FIRST THREE CENTURIES AND THE REFORMATION. THIRD EDITION. 5*s*.

MR. SWAINSON'S (Prebendary of Chichester) HANDBOOK TO BUTLER'S ANALOGY. 1*s*. 6*d*.

MR. WESTCOTT ON THE NEW TESTAMENT CANON. 12*s*. 6*d*.

13